I. QFT in 60 minutes

- Hilbert space $\mathcal{H}$, (ψ, ψ'), $\psi_0 \in \mathcal{H}$ "Vacuum"
- Field operators $V_\alpha(x)$ $\quad x = (t, \vec{x}) \in \mathbb{R} \times \mathbb{R}^{d-1}$

  $\uparrow$ Label $\qquad x^2 = \vec{x}^2 - t^2$
- Wightman functions

$$W_n(x_1 \ldots x_n) := (\psi_0, V_{\alpha_1}(x_1) \ldots V_{\alpha_n}(x_n)\psi_0)$$

- Axioms:
  - space time symmetries
  - regularity
  - Locality $[V_\alpha(x), V_\beta(y)] = 0 \quad (x-y)^2 > 0$

$$V_\alpha(t, \vec{x}) = e^{-itH} V_\alpha(0, \vec{x}) e^{itH}$$

H Hamiltonian $\quad H\psi_0 = 0$

$$W_n = (\psi_0, V_{\alpha_1}(0, \vec{x}_1) e^{i(t_1 - t_2)H} V_{\alpha_2}(0, \vec{x}) e^{i(t_2 - t_3)H} \ldots \psi_0$$

analytic in $\operatorname{Im}(t_i - t_{i+1}) > 0$

Analytic continuation $t \to it$

$$S_n(y_1 \dots y_n) = W_n(x_1, \dots x_n) \qquad x = (iy^1, \vec{y})$$

$$x^2 = \|y\|^2 > 0 \Rightarrow [V_\alpha(y), V_\beta(y')] = 0$$

Euclidean QFT $\quad S_n$ correlation functions of random fields

- Probability space Ω, expectation $\langle - \rangle$
- Random fields $V_\alpha(y, \omega) \quad y \in \mathbb{R}^d, \omega \in \Omega$
- $S_n(x_1, \dots, x_n) = \left\langle \prod_{i=1}^{n} V_\alpha(y_i) \right\rangle$

Axioms (Osterwalder-Schrader)

- symmetries, regularity
- Reflection positivity

$\Rightarrow \quad S_n \longrightarrow W_n \quad$ Reconstruction

Path Integrals

Wightman axioms were motivated by QED, described by an Action functional

- Fields $\varphi : \mathbb{R}^d \to M$ "target space"
- Action $S(\varphi) = \int_{\mathbb{R}^d} \mathcal{L}(\varphi, x)\, dx$
- Locality $\mathcal{L}(\varphi, x) = L(\varphi(x), \nabla\varphi(x), \dots)$
- $V_\alpha(x) = \tilde{V}_\alpha(\varphi(x), \nabla\varphi(x), \dots)$

Feynman:

$$W_n(x_1, \dots, x_n) = \int e^{\frac{i}{\hbar} S(\varphi)} \prod V_\alpha(x_i)\, D\varphi$$

$$\xrightarrow[t \to it]{} \int e^{-\frac{1}{\hbar} S_{EUCL}(\varphi)} \prod_i V_\alpha(x_i)\, D\varphi$$

<u>Example</u> φ^4 theory $\varphi \in \mathbb{R}$

$$\mathcal{L} = -(\partial_t \varphi)^2 + (\nabla\varphi)^2 + m^2\varphi^2 + \lambda\varphi^4$$

$$\mathcal{L}_{eucl} = (\partial_t \varphi)^2 + (\nabla\varphi)^2 + m^2\varphi^2 + \lambda\varphi^4$$

V_α: $\varphi(x)^n$, $(\nabla\varphi)^2$, $e^{i\alpha\varphi}$ ....

Constructive QFT (60's) : make sense of $e^{-S(\varphi)}\,D\varphi$ as **LAW** of φ

Ex Free field on $\mathbb{R}^d$

$$S(\varphi) = \frac{1}{2}\int_{\mathbb{R}^d} [(\nabla\varphi)^2 + m^2\varphi^2]\,dx$$

$e^{-S(\varphi)}\,D\varphi :=$ Gaussian measure with covariance

$$\langle \varphi(x)\varphi(y)\rangle = (-\Delta + m^2)^{-1}(x,y)$$
$$\sim |x-y|^{2-d} \quad x \to y$$

φ distribution valued

$$\varphi \in H^{-\frac{d-2}{2}-\varepsilon}(\mathbb{R}^d) \quad \text{a.s.}$$

$\Rightarrow \quad \varphi(x)^4$ not defined

Regularize : $\varphi \to \chi_\varepsilon * \varphi$ $\quad \varepsilon$ "UV cutoff"

Renormalize

$$\lambda\varphi^4 \to (\lambda\varphi^4)_\varepsilon = \lambda(\chi_\varepsilon * \varphi)^4 + \text{"counter term"}$$

Prove $\quad \frac{1}{Z_\varepsilon} e^{-\lambda\int(\lambda\varphi^4)_\varepsilon}\,\mathbb{P}_{GFF} \xrightarrow{\varepsilon\to 0} \mathbb{P}_\lambda$

Glimm, Jaffe 60-70's: for $d=2,3$

$$(\lambda\varphi^4)_\varepsilon = \lambda\varphi_\varepsilon^4 + a_\varepsilon\varphi_\varepsilon^2$$

$$a_\varepsilon = \begin{cases} \lambda\, a \log\varepsilon & d=2 \\ \lambda\, b\,\varepsilon^{-1} + \lambda^2 c \log\varepsilon & d=3 \end{cases}$$

Physics $d=4$ renormalizable, like QED, QCD

$$(\lambda\varphi^4)_\varepsilon = g_\varepsilon\varphi_\varepsilon^4 + a_\varepsilon\varphi_\varepsilon^2 + b_\varepsilon(\nabla\varphi_\varepsilon)^2$$

$$g_\varepsilon(\lambda) = \lambda + \sum_{n=1}^{\infty}\alpha_n(\varepsilon)\lambda^n, \quad a_\varepsilon(\lambda),\ b_\varepsilon(\lambda) \text{ idem}$$

$$\alpha_n(\varepsilon) \to \infty \qquad \varepsilon\to 0$$

s.t. $$\langle \prod \varphi(x_i)\rangle_\varepsilon = \sum_{n=0}^{\infty} G_n(\underset{\sim}{x},\varepsilon)\lambda^n$$

where $$\lim_{\varepsilon\to 0} G_n(\underset{\sim}{x},\varepsilon) \text{ exists}$$

- Formal power series

- G_n given by Feynman diagrams

Critical phenomena

QFT Short distance singularities as $\varepsilon \to 0$

$$\langle \varphi(x)\, \varphi(y) \rangle \sim \frac{1}{|x-y|^a}$$

Scale invariance in UV

Phase transitions ε fixed, eg lattice φ^4

$$\langle - \rangle = \frac{1}{Z} \sum_{\varphi} (-) \exp\Big[-\beta \sum_{x \in \varepsilon\mathbb{Z}^d} S(\varphi, x)\Big]$$

Large distances:

$$\langle \varphi(x)\, \varphi(y) \rangle \underset{|x-y|\to\infty}{\sim} \begin{cases} e^{-|x-y|/\xi} & \text{non critical} \\ |x-y|^{-\alpha} & \text{critical} \end{cases}$$

Scale invariance in large scales

Renormalisation Group allows to study both: Vary observation scale

$$\varphi \Rightarrow \{\varphi_\ell\}_{\ell \in \mathbb{R}_+} : \varphi_\ell \text{ has scales} \geq \ell$$

$$\text{E.g } \hat{\varphi}(k),\ |k| < \ell^{-1}$$

$$\mathrm{Law}(\varphi_\ell) \propto e^{-S_\ell(\varphi_\ell)}\, D\varphi_\ell$$

$\ell \to S_\ell$ RG flow

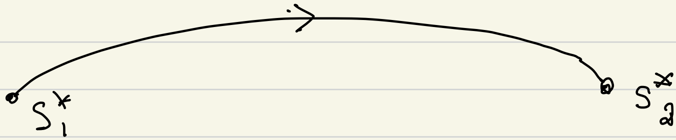
Fixed points: $S_\ell = S^* \quad \forall \ell$

come from scale invariant QFT's

$$\left\langle \prod_{i=1}^{n} V_{\alpha_i}(\lambda x_i) \right\rangle = \lambda^{-2\sum \Delta_{\alpha_i}} \left\langle \prod V_{\alpha_i}(x_i) \right\rangle \qquad (*)$$

Δ_α scaling dimension of V_α

<u>Belief</u> Such QFT are conformally invariant and a generic QFT:



<u>Example</u> φ^4 $d=2,3$ S_1^* = Free field

S_2^* = Ising model

Upshot: Space of all QFT = unstable manifolds of CFT's

Conformal invariance in $d=2$

Conformal group extends Poincaré group by spatially dependent scalings.

In $d=2$ these are Möbius maps ψ

$$z \in \hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\} \to \frac{az+b}{cz+d} \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{C})$$

Axiom 1 2d CFT is postulated to have primary fields $V_\alpha(z)$ s.t.

$$\left\langle \prod_{i=1}^{n} V_{\alpha_i}(\psi(z_i)) \right\rangle = \prod_{i=1}^{n} |\psi'(z_i)|^{-2\Delta\alpha_i} \left\langle \prod_{i=1}^{n} V_{\alpha_i}(z_i) \right\rangle$$

Consequences

1. $\langle V_\alpha(z) \rangle = 0$ or ∞ unless $\Delta_\alpha = 0$

2. If $\langle V_{\alpha_1}(z_1) V_{\alpha_2}(z_2) \rangle \neq \infty$ then

$$\langle V_{\alpha_1}(z_1) V_{\alpha_2}(z_2) \rangle = 0 \quad \text{unless } \Delta_{\alpha_1} = \Delta_{\alpha_2}$$

and $$= \frac{M_{\alpha_1 \alpha_2}}{|z_1 - z_2|^{4\Delta}} \quad \text{if } \Delta_{\alpha_1} = \Delta_{\alpha_2} = \Delta$$

3. $\psi : z_1, z_2, z_3 \to (0, 1, \infty) \Rightarrow$

$$\langle V_{\alpha_1}(z_1) V_{\alpha_2}(z_2) V_{\alpha_3}(z_3) \rangle = C_{\alpha_1 \alpha_2 \alpha_3} *$$

$$* |z_1 - z_2|^{\Delta_{12}} |z_2 - z_3|^{\Delta_{23}} |z_1 - z_3|^{\Delta_{13}}$$

$\Delta_{12} = \Delta_1 + \Delta_2 - \Delta_3$ etc.

$C_{\alpha_1 \alpha_2 \alpha_3}$ "structure constant"

4. $$\langle \prod_{i=1}^{4} V_{\alpha_i}(z_i) \rangle = F(\eta) \prod_{i<j} |z_i - z_j|^{2(\Delta - \Delta_i - \Delta_j)}$$

$$\Delta = \frac{1}{3} \sum \Delta_i , \quad \eta = \frac{(z_1 - z_2)(z_3 - z_4)}{(z_1 - z_3)(z_2 - z_4)}$$

F undetermined function

Operator Product Expansion

Wilson: Short distance expansion in QFT

$$V_\alpha(x) V_\beta(y) \underset{x \to y}{\sim} \sum_\gamma \underbrace{D_{\alpha\beta\gamma}(x-y)}_{\text{Homog degree } 2(\Delta_\alpha + \Delta_\beta - \Delta_\gamma)} V_\gamma(y)$$

CFT : V_α, V_β primary $\Rightarrow V_\gamma = \partial_y^n$ (Primary)

Axiom 2 Let V_α, V_β be primary. Then

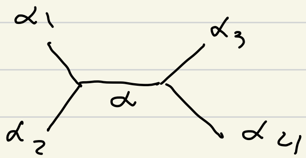
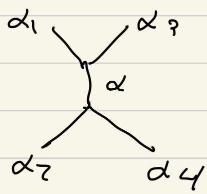
$$V_{\alpha_1}(z_1) V_{\alpha_2}(z_2) = \sum_\alpha C_{\alpha_1 \alpha_2 \alpha} \mathcal{D}_\alpha V_\alpha(z_2)$$

- Convergent in $\langle \prod V_{\alpha_i}(z_i) \rangle$ if $|z_1 - z_2| < |z_i - z_j|$
- $\mathcal{D}_\alpha = \mathcal{D}(z_1, z_2, \underset{\sim}{\alpha}, \partial_{z_2})$ diff. operator
- $\mathcal{D}_\alpha$ determined by conf. symmetry

Consequence $\langle \prod_{i=1}^n V_{\alpha_i}(z_i) \rangle$ iteratively determined in terms of structure constants

Structure constants in turn are constrained by associativity of OPE:

$$\langle V_{\alpha_1}(z_1) V_{\alpha_2}(z_2) | V_{\alpha_3}(z_3) V_{\alpha_4}(z_4) \rangle$$

$=$  $=$  "crossing"

$\Rightarrow$ Quadratic constraint for $C_{\alpha\beta\gamma}$

How to find CFT's?

(1)

- Try to guess the spectrum of primary V_α
- Find solutions to crossing constraints

(2) Construct CFT from a path integral and verify axioms

Conformally invariant actions

Find a conformally invariant action functional and try to construct the path integral

Natural setup: Riemannian geometry

- (Σ, g): Σ 2d surface w. Riemannian metric g
- g, g' conformally equivalent if $g = e^{\sigma} g'$ with $\sigma \in C^\infty(\Sigma)$

Let $\varphi : \Sigma \to M$ (say smooth), M manifold

Local action

$$S(\varphi, g) = \int_\Sigma \mathcal{L}(\varphi, g, x)\, dv_g$$

is conformally invariant if

(i) $S(\varphi, g) = S(\varphi \circ \psi, \psi^* g)$ $\quad \psi \in \mathrm{Diff}(\Sigma)$

(ii) $S(\varphi, e^\sigma g) = S(\varphi, g)$ $\quad \sigma \in C^\infty(\Sigma)$
(Weyl)

Examples

(1) Free field $\quad M = \mathbb{R}$,

$$\mathcal{L} = \frac{1}{4\pi} |d\phi|_g^2 = \frac{1}{4\pi} g^{\alpha\beta} \partial_\alpha \phi \, \partial_\beta \phi$$

$g^{\alpha\beta}$ inverse of $g = g_{\alpha\beta}\, dx^\alpha \otimes dx^\beta$

i.e. $g^{\alpha\beta} g_{\beta\gamma} = \delta^\alpha{}_\gamma \Rightarrow g^{\alpha\beta} \to e^{-\sigma} g^{\alpha\beta}$

$dv_g = \sqrt{\det g_{\cdot\cdot}}\; d^2 x \Rightarrow dv_g \to e^\sigma dv_g$

so Weyl ok.

S is Dirichlet energy and critical point is a harmonic function

(2) σ - Models $\varphi : \Sigma \to M$ where

(M, m) Riemannian, metric m

$d\varphi(x) \in T_x^* M \otimes T_{\varphi(x)} M$

$\| \ \|$ the metric inherited from g, m

$$\mathcal{L}(\varphi, g, x) = \frac{1}{4\pi} \| d\varphi(x) \|^2$$

$$= \frac{1}{4\pi} g^{\alpha\beta} \partial_\alpha \varphi^i(x) \partial_\beta \varphi^j(x) m_{ij}(\varphi(x))$$

$$m(\varphi) = m_{ij} \, d\varphi^i \otimes d\varphi^j$$

Extrema: Harmonic maps $\Sigma \to M$

(3) $P(\varphi)_2$ - model $\varphi : \Sigma \to \mathbb{R}$

$$\mathcal{L} = \frac{1}{4\pi} |d\phi|_g^2 + P(\varphi(x)) \quad \uparrow \text{poly}$$

not Weyl invariant. Likewise sine Gordon $\cos \beta\varphi(x)$ and sinh Gordon $\cosh \beta \varphi(x)$ are not Weyl invariant

(4) Liouville model $\varphi : \Sigma \to \mathbb{R}$

$$\mathcal{L} = \frac{1}{4\pi}\left[\, |d\varphi|_g^2 + Q K_g \varphi + \mu e^{\gamma \varphi} \right] dv_g$$

is (almost):

$$S(\varphi, e^{\sigma} g) = S(\varphi + \sigma, g] - 6Q^2 A(g, \sigma)$$

where Weyl Anomaly:

$$A = \frac{1}{96\pi} \int \left[\, |d\sigma|^2 + 2 R_g \sigma \right] dv_g$$

Here $Q = \frac{2}{\gamma}$, $\mu > 0$ and K_g is scalar curvature.

- Recall: $\exists$ atlas on Σ s.t transition functions are holomorphic and in such coordinates $g = \frac{1}{2} g(z)(dz \otimes d\bar{z} + d\bar{z} \otimes dz)$

Riemannian volume is $dv_g = g(z)\, d^2 z$, and

$$K_g = -4\, \partial_z \partial_{\bar{z}} \log g(z)$$

Minimizer $\varphi = \varphi_g$ s.t $K_{e^{\varphi_g} g} = \text{const}$ for genus$(\Sigma) \geq 2$.

Renormalisation

Try to define

$$\left\langle \prod_{i=1}^{n} V_{\alpha_i}(z_i) \right\rangle = \int_{\varphi : \Sigma \to M} e^{-S(\varphi, g)} \prod V_{\alpha_i}(\varphi, x_i)\, D\varphi$$

as for φ^4_d by regularising and renormalising

Example Heisenberg model = σ-model with $\Sigma = \mathbb{C}$, $M = S^{N-1}$, round metric m. Then RG calculation gives:

Take $S^\varepsilon(\varphi_\varepsilon, m) = S(\varphi_\varepsilon, \frac{1}{T_\varepsilon} m)$. Then

$$\lim_{\varepsilon \to 0} S^\varepsilon_\ell = S(\varphi_\ell, \tfrac{1}{T_\ell} m) + \text{irrelevant}$$

with $\ell \dfrac{dT}{d\ell} = (N-2)T^2 + O(T^3)$

i.e. $T_\ell \sim \dfrac{1}{(N-2)\log \ell^{-1}} \to 0$ as $\ell \to 0$
"asymptotic freedom"

T_ℓ is temperature and this analysis is believed to be valid for T small.

Challenge Give a rigorous proof!

This shows theory is not a CFT. For $l \to 0$ it flows to UV fixed point = free field

For $l \to \infty$ it is believed to flow to the high temperature fixed point $T_l \to \infty$ which is not conformal. Correlations decay exponentially in distance.

Proof of this is even bigger challenge!

<u>Note</u> For $N = 2$ i.e. $M = S^1$ $\quad l \frac{dT}{dl} = 0 + \cdots$

and this theory is conformal, GFF with values in the circle.

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For general σ-model one gets (Friedan, '82)

$$S_l \approx S(\varphi_l, m_l)$$

$$l \frac{d}{dl} m_l = R(m_l) + \cdots$$

$R = R_{ij}\, d\varphi^i \otimes d\varphi^j$ is Ricci tensor of m_l

I.e. RG flow is Ricci flow. Hence possible CFT are Ricci flat M. However corrections ... probably spoil this.

A SuSy version ($N=2$ SuSy) with M Calabi-Yau ($=$ Kähler, Ricci flat) might be deformed to CFT and $N=0$ Hyper-Kähler has $\ell \frac{d}{d\ell} m_\ell = 0$ to all order in the low temperature expansion.

Topological term. One can add to the σ-model action a term of form

$$S_{TOP} = \int_\Sigma B_{ij}(\varphi(x))\, d\varphi^i(x) \wedge d\varphi^j(x)$$

where $B = B_{ij}\, d\varphi^i \wedge d\varphi^j$ is a 2-form on M

This way one gets a CFT, the WZW-model we'll discuss later.

2. Local conformal invariance

Let us formulate CFT in the Riemannian setup. We suppose primary field correlation functions are defined and satisfy

(1) Diff Let $\psi \in \mathcal{D}iff(\Sigma)$ (smooth)

$$\langle \prod V_{\alpha_i}(z_i) \rangle_g = \langle \prod V_{\alpha_i}(\psi(z_i)) \rangle_{\psi^* g}$$

(2) Weyl Let $\sigma \in C^\infty(\Sigma)$. Then

$$\langle \prod V_{\alpha_i}(z_i) \rangle_{e^\sigma g} = e^{cA(g,\sigma)} \prod_i e^{-\Delta_{\alpha_i}\sigma(z_i)} \langle \prod V_{\alpha_i}(z_i) \rangle_g$$

where, as before, $A = \frac{1}{96\pi}\int [|\partial\sigma|^2 + 2K_g\sigma]\, dv_g$

is the Weyl anomaly and c is the central charge of the CFT

Note. Let $\Sigma = \hat{\mathbb{C}}$, ψ Möbius. Then $\psi^* g = |\psi'|^2 g\circ\psi$

so using (1), (2) we recover old Axiom I since in this case $A(g,\sigma) = 0$ (Check!).

Stress - Energy tensor (SET)

SET describes variation of a local action functional $S(\phi, g)$ in the metric:

$$T_{\mu\nu}(x) = \frac{\delta}{\delta g^{\mu\nu}(x)} S(\phi, g)$$

i.e. vary the inverse metric

$$g_\varepsilon^{\mu\nu} = g^{\mu\nu} + \varepsilon f^{\mu\nu}$$

where $f^{\mu\nu}$ is smooth and then

$$\frac{d}{d\varepsilon}\Big|_0 S(\phi, g) = \int_\Sigma T_{\mu\nu}(x)\, f^{\mu\nu}(x)\, dv_g(x)$$

Example For free field we get

$$T_{\mu\nu} = \partial_\mu \phi\, \partial_\nu \phi - g_{\mu\nu}\, g^{\alpha\beta}\, \partial_\alpha \phi\, \partial_\beta \phi$$

(second term comes from dv_g)

Exercise: compute $T_{\mu\nu}$ for Liouville!

Conformal Ward Identities

In the quantum case we study variations of the correlation functions $\langle \prod_{i=1}^{n} V_{\alpha_i}(z_i) \rangle_{\Sigma, g}$ when we vary the metric g. Let again

$$g_\varepsilon^{\mu\nu} = g^{\mu\nu} + \varepsilon f^{\mu\nu}$$

where f is smooth with compact support in the complement of the points $\{z_i\}$.

We try to define

$$\frac{d}{d\varepsilon}\Big|_0 \langle \prod V_{\alpha_i}(z_i) \rangle_{\Sigma, g_\varepsilon} = \int_\Sigma \langle T_{\mu\nu}(z) \prod V(z_i) \rangle f^{\mu\nu}(z)\, dv_g(z)$$

The axioms Diff & Weyl allow us to determine such variation explicitely if $\Sigma = \hat{\mathbb{C}}$.

Proposition Assume

$$F(\underset{\sim}{z}, g) := \langle \prod_{i=1}^{n} V_{\alpha_i}(z_i) \rangle_{\hat{\mathbb{C}}, g}$$

is smooth in $\underset{\sim}{z}$ in the region $z_i \neq z_j \ \forall i \neq j$ and satisfies Diff & Weyl. Then

$\varepsilon \to F(\underset{\sim}{z}, g_\varepsilon)$ is smooth around $\varepsilon = 0$ and

$$\frac{d}{d\varepsilon}\Big|_{\varepsilon=0} F(\underset{\sim}{z}, g_\varepsilon) = \int f^{\mu\nu}(z)\, F_{\mu\nu}(z, \underset{\sim}{z}, g)\, dv_g(z)$$

where $F_{\mu\nu}(z, \underset{\sim}{z}, g)$ is smooth in $z \in \hat{\mathbb{C}} \setminus \cup z_i$

We denote

$$F_{\mu\nu}(z, \underset{\sim}{z}, g) = \langle T_{\mu\nu}(z) \prod V_{\alpha_i}(z_i) \rangle_g$$

Idea of proof $\hat{\mathbb{C}}$ has one conformal class so we take $g = e^{\sigma} |dz|^2$.

By Weyl & Diff we get $\exists\, \psi_\varepsilon \in \mathcal{D}iff$, $\sigma_\varepsilon \in C^\infty(\hat{\mathbb{C}})$

$$g_\varepsilon = e^{\sigma_\varepsilon} \psi_\varepsilon^* g$$

Then

$$\frac{d}{d\varepsilon}\Big|_0 \langle \prod V_{\alpha_i}(z_i) \rangle_{\hat{\mathbb{C}}, g_\varepsilon} = \frac{d}{d\varepsilon}\Big|_0 \Big[e^{-cA(\psi_\varepsilon^* g, \sigma_\varepsilon)} \prod e^{-\Delta_\alpha \sigma_\varepsilon(z_i)}$$

$$* \langle \prod V_{\alpha_i}(\psi_\varepsilon(z_i)) \rangle_g \Big] = \mathcal{D}_{\underset{\sim}{z}} \langle \prod V_{\alpha_i}(z_i) \rangle_{\hat{\mathbb{C}}, g}$$

where $\mathcal{D}_{\underset{\sim}{z}}$ is a linear differential operator.

To find ψ_ε, σ_ε one needs to solve a Beltrami equation

$$\partial_{\bar z} \psi_\varepsilon = \mu \, \partial_z \psi_\varepsilon$$

where $\mu = \eta_{z\bar z} \left[(1 + 4 \eta_{z\bar z})^2 - 4 \eta_{zz} \eta_{\bar z \bar z} \right]^{-1}$

where $g_\varepsilon = g + e^{\sigma} \eta$.

Put $\psi_\varepsilon(z) = z + u_\varepsilon(z)$. Then

$$\partial_{\bar z} u - \mu \partial_z u = \mu \qquad (*)$$

$\partial_{\bar z}^{-1}$ is given by Cauchy transform $\mathcal{C}$

$$(\mathcal{C} f)(z) = \frac{1}{\pi} \int_{\mathbb{C}} \frac{f(v)}{z - v} \, d^2 v$$

so that $(*)$ becomes $(1 - \mathcal{C}(\mu \partial_z))^{-1} u = \mathcal{C} \mu \Rightarrow$

$$u = (1 - \mathcal{C} \mu \partial_z)^{-1} \mathcal{C} \mu = \mathcal{C} \sum_{n=0}^{\infty} (\mu \mathcal{B})^n \mu$$

where $\mathcal{B} = \partial_z \mathcal{C} = \mathcal{C} \partial_z$ is Beltrami transform

Upshot: The solution is C^∞ in ε and $\frac{d}{d\varepsilon} \Gamma_0$ can be explicitely computed.

We can write the tensor $T_{\mu\nu}\, dx^\mu \otimes dx^\nu$ in the $z, \bar{z}$ basis:

$$T = T_{zz}\, dz^2 + T_{\bar{z}\bar{z}}\, d\bar{z}^2 + T_{z\bar{z}}\, (dz\, d\bar{z} + d\bar{z}\, dz)$$

Then

$$T_{zz}(z) = \frac{1}{4}\left(T_{11}(z) - T_{22}(z) - 2i\, T_{12}(z) \right)$$

where 1,2 refer to $z = x_1 + i x_2$. Define

$$T(z) = T_{zz}(z) + \frac{c}{12}\, t(z)$$

$$t(z) = \partial_z^2 \sigma(z) - \frac{1}{2}\left(\partial_z \sigma(z)\right)^2$$

Calculation gives

$$\langle T(z) \prod V_{\alpha_i}(z_i) \rangle = \sum_{j=1}^{n} \left(\frac{\Delta_{\alpha_j}}{(z-z_j)^2} + \frac{\Delta_{\alpha_j}\, \partial_z \sigma(z_j)}{(z-z_j)} \right.$$

$$\left. + \frac{1}{z-z_j}\, \partial_{z_j} \right) \langle \prod_i V_{\alpha_i}(z_i) \rangle$$

This is called conformal Ward identity.

Note that $T(z)$ is holomorphic in $\hat{\mathbb{C}} \setminus \{z_i\}$.

Next, take $g_{\varepsilon_1\varepsilon_2} = g + \varepsilon_1 f_1 + \varepsilon_2 f_2$ with

f_i are supported around $u, v \in \{z_i\}$. Get:

$$\langle T(u)T(v)\prod V_{\alpha_i}(z_i)\rangle = \Big\{ \frac{c/2}{(u-v)^4} + \frac{2}{(u-v)^2} + \frac{1}{u-v}\partial_v$$

$$+ \sum \Big(\frac{\Delta\alpha_i}{(u-z_i)^2} + \frac{1}{u-z_i}\partial_{z_i} \Big) \langle T(v)\prod V_{\alpha_i}(z_i)\rangle$$

Similar expressions for $\langle \prod_j T(u_j) \prod V_{\alpha_i}(z_i)\rangle$

Let C_i be a contour surrounding z_i and no other z_j. Let

$$L_n(z_i) = \frac{1}{2\pi i}\oint_{C_i} (z-z_i)^{n+1}\, T(z)$$

Then

$$\langle L_n V_{\alpha_i}(z_i) \prod_{j\neq i} V_{\alpha_j}(z_j)\rangle = \begin{cases} 0 & n>0 \\ \Delta_{\alpha_i}\langle \prod V_{\alpha_i}\rangle & n=0 \\ \sum_{j\neq i}\Big(-\frac{1}{(z_j-z_i)^{n-1}}\frac{\partial}{\partial z_j} + \frac{n-1}{(z_j-z_i)^n}\Delta_j\Big) \\ \quad * \langle \prod V_{\alpha_i}\rangle & n<0 \end{cases}$$

So $L_n V_\alpha = 0$ $n>0$, $L_0 V_\alpha = \Delta_\alpha V_\alpha$

Furthermore, a nice excercise:

$$[L_n(z_i), L_m(z_i)] = (n-m) L_{n+m} + \frac{c_L}{12}(n^3-n)\delta_{n,-m}$$

Similar story with $\tilde{T}(z) = T_{\bar{z}\bar{z}}$.

$\{L_n\}$, $\{\bar{L}_n\}$ form two commuting

Virasoro algebras.

Let $\nu = (\nu_1, \nu_2, \ldots \nu_k)$ with $\nu_i \in \mathbb{Z}_-$

$$\nu_k \leq \nu_{k-1} \leq \ldots \leq \nu_1 < 0 \qquad k < \infty$$

Define $L_\nu = L_{\nu_k} L_{\nu_{k-1}} \cdots L_{\nu_1}$, $\bar{L}_{\bar{\nu}}$ similarly

The fields $V_{\alpha,\nu,\bar{\nu}} = L_\nu \bar{L}_{\bar{\nu}} V_\alpha$ are decendants of V_α

Ward $\Rightarrow$ holomorphic factorisation

$$\left\langle \prod_{i=1}^{n} V_{\alpha_i \nu_i \tilde{\nu}_i}(z_i) \right\rangle_{\hat{C},g} = \mathcal{D}(\underset{\sim}{\Delta}_\alpha, \underset{\sim}{\nu})\, \overline{\mathcal{D}(\underset{\sim}{\Delta}_\alpha, \underset{\sim}{\tilde{\nu}})} \left\langle \prod V_{\alpha_i}(z_i) \right\rangle$$

$\mathcal{D}$ diff operator in $\{\frac{\partial}{\partial z_i}\}$ determined by

c, $\{\Delta_{\alpha_i}\}$, $\{z_i\}$

This is crucial for conformal bootstrap.

2. Probabilistic LCFT

We want to define the path integral

$$\langle F(\varphi)\rangle_{\Sigma,g} = \int F(\varphi)\, e^{-S(\varphi,g)}\, D\varphi \qquad (*)$$

for Liouville action functional

$$S(\varphi,g) = \frac{1}{4\pi}\int_\Sigma \left[|d\varphi|_g^2 + Q K_g \varphi + 4\pi\mu e^{\gamma\varphi} \right] dv_g$$

Recall that this action is Diff invariant and under Weyl transforms as

$$S(\varphi, e^{\sigma} g) = S(\varphi+\sigma, g) - 6Q^2 A(g,\sigma)$$

We define $(*)$ as a Radon-Nikodym derivative w.r.t. the free field, $Q=\mu=0$.

Free field. The Dirichlet energy for a smooth φ can be written as

$$\frac{1}{4\pi}\int_\Sigma |d\varphi|_g^2\, dv_g = \frac{1}{4\pi}\left(\varphi, (-\Delta_g)\varphi\right)_{L^2(\Sigma, dv_g)}$$

where the Laplace-Beltrami operator

$\Delta_g = \partial_\alpha (g^{\alpha\beta} \sqrt{\det g}\, \partial_\beta)$ is self adjoint on $L^2(\Sigma, dv_g)$. Moreover $-\Delta_g$ has a discrete spectrum of eigenvalues $\{\lambda_{g,n}\}$

a) $-\Delta_g e_{g,n} = \lambda_{g,n} e_{g,n} \qquad n = 0, 1, \ldots$

(b) $\lambda_{g,n} > 0 \quad n > 0, \ \lambda_{g,0} = 0, \ e_{g,0} = \text{constant}$

(c) $(e_{g,n}, e_{g,n'}) = \delta_{n,n'}$

Expand φ in this basis

$$\varphi(x) = c + \sqrt{2\pi} \sum_{n=1}^{\infty} \frac{a_n}{\sqrt{\lambda_{g,n}}} e_{g,n}(x)$$

$$\Rightarrow \quad e^{-\frac{1}{4\pi}\int_\Sigma |d\varphi|_g^2 \, dv_g} = e^{-\frac{1}{2}\sum_{n=1}^{\infty} a_n^2}$$

So we can define the free field path integral in terms of a variable $c \in \mathbb{R}$, and a product of independent normal variables $\{a_n\}_{n=1}^{\infty}$

Gaussian Free Field (GFF) on (Σ, g):

$$X_g(x) = \sqrt{2\pi} \sum_{n=1}^{\infty} \frac{a_n}{\sqrt{\lambda_{g,n}}} e_{g,n}(x)$$

where a_n i.i.d. $N(0,1)$.

Let

$$H^s(\Sigma, g) = \left\{ \sum_{n=0}^{\infty} \varphi_n e_n \;\middle|\; \sum |\varphi_n|^2 (1+n)^{2s} < \infty \right\}$$

Since $\lambda_{g,n} \sim c\, n$ as $n \to \infty$ we have

$$\mathbb{E}\left(\sum_n \frac{a_n}{\sqrt{\lambda_{g,n}}} (1+n)^s \right)^2 \leq c \sum \frac{1}{n^{1-2s}} < \infty$$

if $s < 0$. Hence $X_g \in H^s$, $s<0$ a.s.

Easy excercise:

$$\mathbb{E}\, \phi_g(x) \phi_g(y) = G_g(x,y)$$

$$-\Delta_g G_g(x,y) = \frac{1}{\sqrt{\det g}} \delta(x-y) - \frac{1}{v_g(x)}$$

Fact: $G_g(x,y) = \frac{1}{2\pi} \log d_g(x,y)^{-1} + \text{smooth}$

as $x \to y$.

Free field $\quad \varphi(z) = c + X_g(z) \quad c \in \mathbb{R}$

$$e^{-\frac{1}{4\pi}\int_\Sigma |d\varphi|_g^2 \, dv_g} D\varphi := Z_{\Sigma,g} \, \mathbb{P}(dX_g)\, dc =: \nu^0_{\Sigma,g}(d\varphi) \quad (*)$$

$$Z_{\Sigma,g} = \text{“} \frac{1}{(\det(-\Delta_g))^{1/2}} \text{”} = e^{\frac{1}{2}\zeta'_{\Sigma,g}(0)} \, v_g(\Sigma)^{1/2}$$

"partition function of GFF"

$$\zeta_{\Sigma,g}(s) = \sum_{n=1}^{\infty} \lambda_{g,n}^{s}$$

Note Gaussian measure on $\phi \in \mathbb{R}^N$ with covariance $\mathbb{E}\, \phi_i \phi_j = (A^{-1})_{ij}$ is given by

$$\mathbb{P}(d\phi) = \det(2\pi A)^{1/2} e^{-\frac{1}{2}(\phi, A\phi)} d^N\phi$$

Let A have eigenvalues λ_i. Then

$$\log\det A = \sum \log \lambda_n = \frac{d}{ds}\Big|_{s=0} \sum_n \lambda_n^{s}$$

In our case, $\zeta_{\Sigma,g}$ is analytic in s in $\operatorname{Im} s < -1$ and has a meromorphic continuation to $s \in \mathbb{C}$, with no pole at $s=0$

This motivates $(*)$

Gaussian Multiplicative Chaos GMC

Hence we want to have

$$e^{-S(\varphi,g)} D\phi = \exp\left[-\frac{1}{4\pi}\int_\Sigma (Q K_g \varphi + 4\pi\mu e^{\gamma\varphi})\, dv_g\right] \nu^0_{\Sigma,g}(d\varphi)$$

Since K_g is smooth $\int K_g \varphi \, dv_g$ is a well defined random variable. But $e^{\gamma\varphi}$ is not as

$$\mathbb{E}\, e^{\gamma \varphi(x)} = e^{\gamma^2/2\, G_g(x,x)} = \infty .$$

We regularise

$$X_{g,\varepsilon}(x) := \frac{1}{l_g(C_\varepsilon(x))} \int_{C_\varepsilon(x)} X_g(y)\, d l_g(y)$$

where $C_\varepsilon(x) =$ geodesic circle of radius ε

and renormalise $\mu \to \mu\, \varepsilon^{\gamma^2/2}$.

Prop The random measure $\varepsilon^{\frac{\gamma^2}{2}} \cdot e^{\gamma X_{g,\varepsilon}(x)} dv_g(x)$ converges in the sense of weak convergence of measures and in probability

$$\varepsilon^{\frac{\gamma^2}{2}} \cdot e^{\gamma X_{g,\varepsilon}(x)} dv_g(x) \xrightarrow[\varepsilon\to 0]{} M_{g,\gamma}(dx)$$

The limit $M_{g,\gamma}$ is the GMC measure, it is $\neq 0$ iff $\gamma < 2$ and $M_{g,\gamma}(\Sigma) < \infty$ a.s.

Def. LCFT measure is defined as

$$\nu_{\Sigma,g} = e^{-\frac{Q}{4\pi}\int R_g \varphi \, dv_g - \mu e^{\gamma c} M_{g,\gamma}(\Sigma)} \nu^{\circ}_{\Sigma,g}$$

where Q is renormalised to

$$Q = \frac{\gamma}{2} + \frac{2}{\gamma}$$

We denote, for $F : H^{-s} \to \mathbb{C}$

$$\langle F \rangle_{\Sigma,g} := \int F(\varphi) \, d\nu_{\Sigma,g}(\varphi)$$

provided $\int |F| \, d\nu < \infty$.

Vertex operators Let $\alpha \in \mathbb{R}$

$$V_{\alpha,g,\varepsilon}(x) = \varepsilon^{\alpha^2/2} e^{\alpha(c + \phi_{g,\varepsilon}(x))}$$

$$\left\langle \prod_{i=1}^{n} V_{\alpha_i,g}(x_i) \right\rangle_{\Sigma,g} := \lim_{\varepsilon \to 0} \left\langle \prod_{i=1}^{n} V_{\alpha_i,g,\varepsilon}(x_i) \right\rangle_{\Sigma,g}$$

Prop. The limit exists and is non-trivial provided the Seiberg bounds hold:

$$\sum_{i=1}^{n} \alpha_i - Q\chi(\Sigma) > 0 \quad \& \quad \alpha_i < Q \quad \forall i$$

where $\chi(\Sigma) = (2 - \text{genus})$ is Euler character

The limit satisfies Diff & Weyl with

$$\Delta_\alpha = \frac{\alpha}{2}\left(Q - \frac{\alpha}{2}\right)$$

$$c = 1 + 6Q^2$$

Ideas Since $\int R_g \, dv_g = 4\pi \chi(\Sigma)$

the c-integral converges

$$\int_{\mathbb{R}} e^{(\Sigma\alpha_i - 2Q\chi(\Sigma))c - \mu e^{\gamma c} M_{g,\gamma}(\Sigma)}$$

$$= \mu^{-s} \Gamma(s) \left(M_{g,\gamma}(\Sigma)\right)^{-s/\gamma}$$

if $s = \Sigma\alpha_i - Q\chi(\Sigma) > 0$. Then

$$\left\langle \prod_{i=1}^{n} V_{\alpha_i,g}(z_i) \right\rangle_{\Sigma,g} = \mu^{-s}\,\Gamma(s)\, Z_{\Sigma,g} \lim_{\varepsilon\to 0} \times$$

$$\times\ \mathbb{E} \prod \varepsilon^{\alpha_i^2/2}\, e^{\alpha_i X_{g,\varepsilon}(z_i)} \left(M_{g,\gamma}(\Sigma)\right)^{-s/\gamma}$$

Shift $X_g \to X_g + \sum_i \alpha_i G_g(z_i,\cdot) \Rightarrow$

$$= \mu^{-s}\,\Gamma(s)\, Z_{\Sigma,g}\, C_g(\underset{\sim}{z})\, \mathbb{E}\left(\int e^{\sum_i \gamma\alpha_i G(z_i,z)}\, dM_g(z)\right)^{-s/\gamma}$$

with $C_g(\underset{\sim}{z})$ explicit. Now

$$e^{\gamma\alpha_i G(z_i,z)} \sim \frac{1}{d(z_i,z)^{\gamma\alpha_i}} \quad \text{as } z_i \to z$$

Main Lemma This is GMC integrable if $\alpha_i < Q$ (i.e. $\gamma\alpha_i < 2 + \gamma^2/2$)

Diff follows from

$$X_g \circ \psi \overset{\text{LAW}}{=} X_{\psi^* g}$$

Weyl follows from

1° $X_{g'} \overset{\text{LAW}}{=} X_g - \dfrac{\int X_g \, dv_{g'}}{\int dv_{g'}}$ $\qquad g' = e^{\sigma} g$

$$\Rightarrow \int F(X_{g'} + c)\, dc = \int F(X_g + c)$$

2° $Q \int X_g K_{g'} \, dv_{g'} = Q \int X_g (K_g + \Delta_g \sigma)\, dv_g$

3° $\varepsilon^{\frac{\gamma^2}{2}} e^{\gamma (X_g)_{g', \varepsilon}} \simeq \varepsilon^{\frac{\gamma^2}{2}} e^{\gamma (X_g)_{g, e^{-\sigma}\varepsilon}}$

$$= e^{\gamma^2/2\, \sigma} (\varepsilon')^{\gamma^2/2} e^{\gamma X_{g, \varepsilon'}}$$

$$\Rightarrow dM_{e^{\sigma} g} = e^{2\sigma} e^{\frac{\gamma^2}{2}\sigma} dM_g = e^{\gamma Q \sigma} dM_g$$

$$V_{\alpha, g'} = e^{\frac{\alpha^2}{2}\sigma} V_{\alpha, g}$$

4° Shift $X_g \to X_g - Q\left(\sigma - \dfrac{\int \sigma \, dv_g}{\int dv_g}\right)$

$$\Rightarrow V_\alpha \to e^{-\alpha Q} V_\alpha$$

$$\Rightarrow \Delta_\alpha = \frac{\alpha}{2}\left(Q - \frac{\alpha}{2}\right)$$

5° central change get $6Q^2$ from curvature term & 1 from $\det \Delta_g$ term.

3. Wess - Zumino - Witten model

Let the target manifold be now a compact, semisimple Lie Group G. Let $\mathfrak{g}$ be the Lie Algebra. We only need below $G = SU(2) =$ 2×2 unitary matrices, $\mathfrak{g} = 2\times 2$ anti hermitean matrices. $\exp : \mathfrak{g} \to G$ given by $a \in \mathfrak{g} \to e^{a}$.

G has a natural Riemannian metric m which is Left & Right invariant:

$R_g : G \to G$ $\quad R_g h := h g^{-1}$, $L_g : G \to G$, $L_g h = g h$

$R_g^* m = m = L_g^* m$. m is called Killing form and given by minus $\times$ trace in the adjoint representation of $\mathfrak{g}$. For $SU(2)$ we normalize it as $(a, b) = -\mathrm{Tr}\, a b$ for $a, b \in su(2) =$ Lie Alg. of $SU(2)$.,