Solvability of Schramm-Loewner Evolution via Liouville Quantum Gravity, Lectures I and II

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Summer School on Probabilistic Methods in QFT
Hausdorff Research Institute for Mathematics, May 2025

Outline

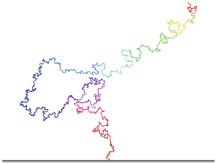
- **What is Schramm-Loewner evolution?**
- Two solvability results linking SLE and conformal field theory.
- Gaussian free field and Gaussian multiplicative chaos.
- Introduction to Liouville quantum gravity.

What is Schramm-Loewner evolution SLE_κ?

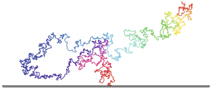
SLE is a random non-self-crossing fractal curve.

It connects two boundary points of a simply connected domain.

"Roughness" parameter κ: $\dim_{\text{Hausdorff}}(\mathrm{SLE}_\kappa) = \min(1 + \frac{\kappa}{8}, 2)$.



SLE_3



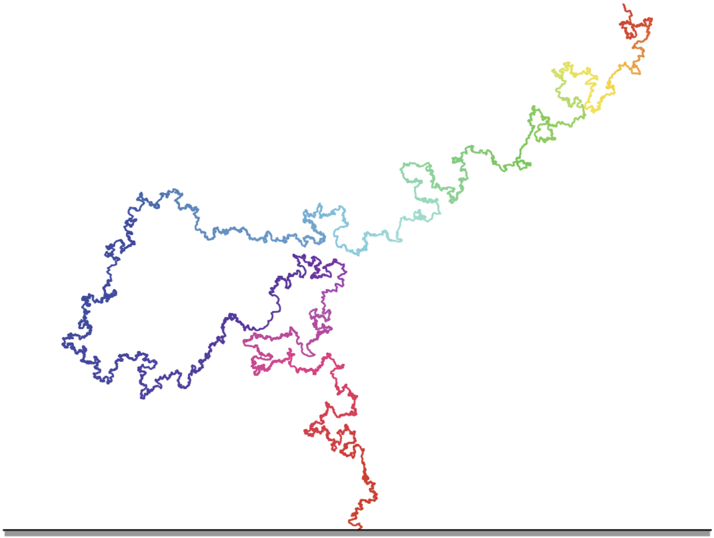
SLE_6

SLE_{32}

Images by Antti Kemppainen

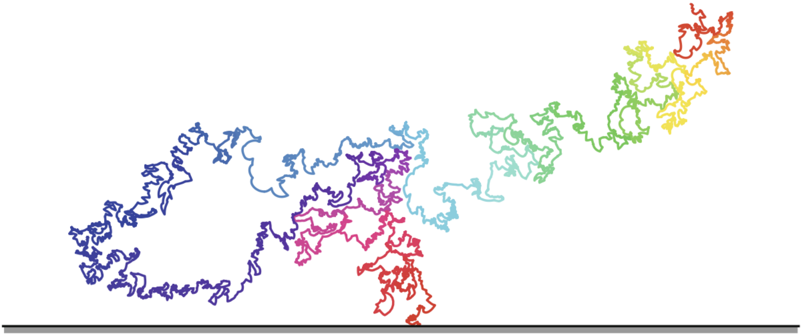
SLE_κ phases:
simple ($\kappa \leq 4$), self-hitting ($4 < \kappa < 8$), space-filling ($\kappa \geq 8$).

Schramm-Loewner evolution with $\kappa = 3$



Simple phase: $\kappa \leq 4$.

Schramm-Loewner evolution with $\kappa = 6$



Self-hitting phase: $4 < \kappa < 8$.

Schramm-Loewner evolution with $\kappa = 32$

Space-filling phase: $\kappa > 8$.

SLE$_\kappa$ = scaling limit of statistical physics interfaces

Critical percolation on the triangular lattice

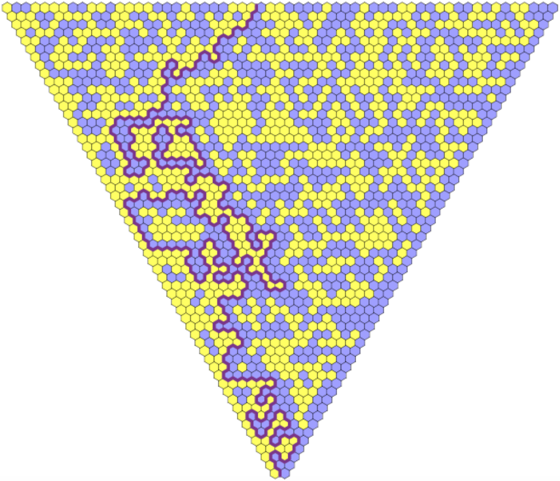


Image by Duminil-Copin

SLE$_6$ $\longleftrightarrow$ Percolation [Smirnov '01]

SLE_κ = scaling limit of statistical physics interfaces

Critical Ising model on the square lattice

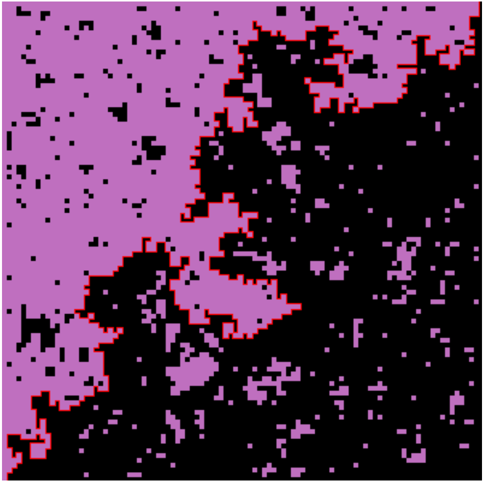


Image by Peltola

$SLE_3 \longleftrightarrow$ Ising model [Smirnov '07]

SLE$_\kappa$ = scaling limit of statistical physics interfaces

Uniform spanning tree with Dobrushin boundary conditions

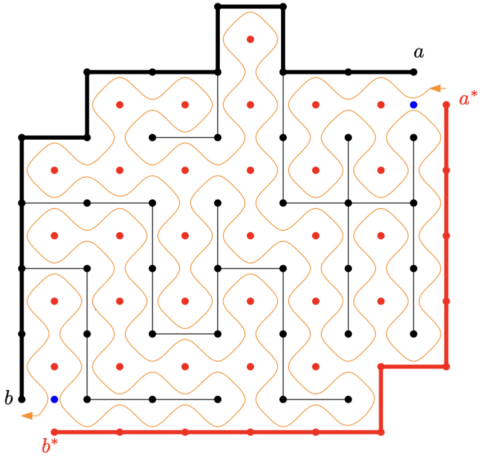


Image from Yong Han-Mingchang Liu-Hao Wu '20.

SLE$_8$ $\longleftrightarrow$ Uniform spanning tree [Lawler-Schramm-Werner '01]

Not a millenium prize problem, but you will become famous

Simply-connected lattice domain with two boundary points.

Consider set of all simple lattice paths between the two points.

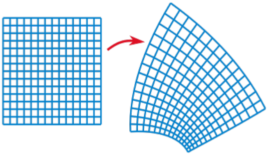
Self-avoiding walk = sample from this set weighted by $c^{-\text{length}}$, where c is the connectivity constant of the lattice.

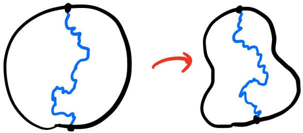
Does self-avoiding walk converge to SLE_κ with $\kappa = 8/3$?

Let SLE' $=$ scaling limit of critical statistical physics interface.

Conformal invariance (in law).

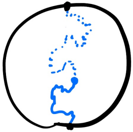
SLE' in $D_2 \stackrel{d}{=}$ SLE' in D_1 under conformal map $D_1 \to D_2$.





Domain Markov property.

Conditioned on SLE' curve run until stopping time,
remainder of curve has the law of SLE' in complement.

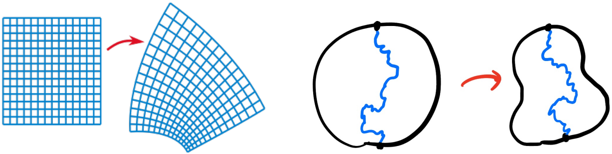


Conjectural properties of scaling limits of discrete models

Let SLE' = scaling limit of critical statistical physics interface.

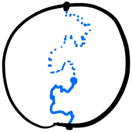
Conformal invariance (in law).

SLE' in $D_2 \overset{d}{=}$ SLE' in D_1 under conformal map $D_1 \to D_2$.



Domain Markov property.

Conditioned on SLE' curve run until stopping time, remainder of curve has the law of SLE' in complement.



Goal: Identify all random curves satisfying these properties.

Rigorous definition of SLE

Consider curve in upper half-plane from 0 to ∞.

Schramm '01: definition of **Stochastic Loewner evolution** via SDE

$$dg_t(z) = \frac{2}{g_t(z) - \sqrt{\kappa} B_t} dt.$$

We will not explain this stochastic differential equation today.

Brownian motion B_t is the only random continuous process satisfying **scale invariance** and **strong Markov property**,

so SLE is the only curve satisfying **conformal invariance** and **domain Markov property**.

Tools to study SLE_κ for general κ

Loosely speaking, three approaches.

- SDE definition

$$dg_t(z) = \frac{2}{g_t(z) - \sqrt{\kappa}B_t}dt.$$

  Use martingales, Itô calculus, etc.

- Coupling with random generalized function called **Gaussian free field** (GFF).

- Coupling with **Liouville quantum gravity**.

We will focus on third approach, and how it allows us to prove predictions from CFT.

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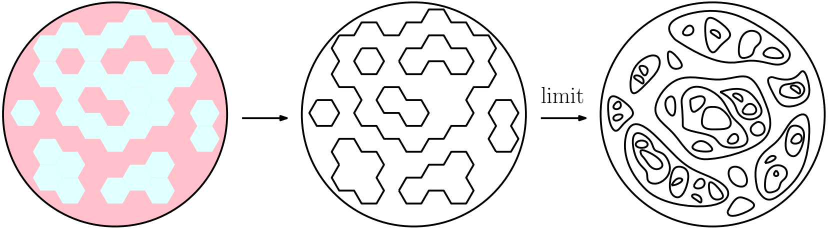
1. Conformal loop ensemble (CLE)

For $\kappa \in (8/3, 8)$, CLE$_\kappa$ is a random collection of non-crossing loops which each locally look like SLE$_\kappa$.

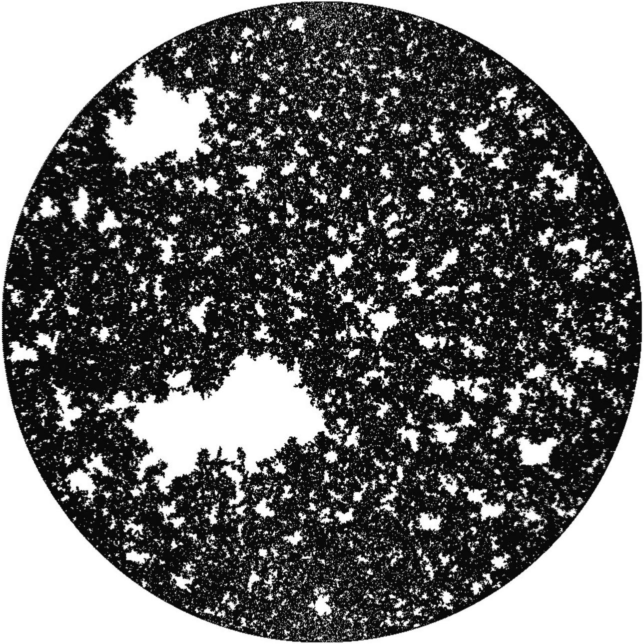
Arises from scaling limits of lattice model interfaces.

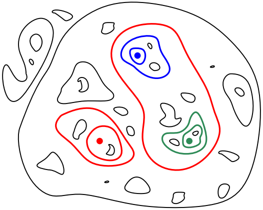


There is a canonical notion of CLE for the disk and sphere.

Conformally invariant in law.

1. Outermost CLE_4 loops in disk (image by David Wilson)

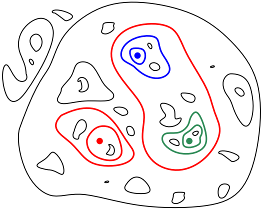




Let $\kappa \in (8/3, 8)$ and $n = -2\cos(4\pi/\kappa) \in (0, 2]$.

Let $z_1, z_2, z_3 \in \mathbb{C}$ and
$X_\varepsilon(z_1) = \#$ loops separating ball $B_\varepsilon(z_1)$ from z_2, z_3.

1. Three point nesting function for CLE on the sphere



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Let $z_1, z_2, z_3 \in \mathbb{C}$ and
$X_\varepsilon(z_1) = \#$ loops separating ball $B_\varepsilon(z_1)$ from z_2, z_3.

Let $n_1, n_2, n_3 \in (0, 2]$ and $\Delta_i = \Delta_i(n_i, \kappa)$ (formula omitted).

$$\lim_{\varepsilon \to 0} \mathbb{E}[\prod_{i=1}^{3} \varepsilon^{-\Delta_i} (\frac{n_i}{n})^{X_\varepsilon(z_i)}]$$
$$= \left(\prod_{i=1}^{3} |z_i - z_{i+1}|^{-\Delta_i - \Delta_{i+1} + \Delta_{i+2}} \right) C_\kappa(\Delta_1, \Delta_2, \Delta_3).$$

1. CLE nesting structure constant = imaginary DOZZ

Reparametrize $\widehat{\alpha}_i = \widehat{\alpha}_i(\Delta_i, \kappa)$ (formula omitted).

$$R_\kappa^{\text{nest}}(\widehat{\alpha}_1, \widehat{\alpha}_2, \widehat{\alpha}_3) := \frac{C_\kappa(\Delta_1, \Delta_2, \Delta_3)}{\sqrt{C_\kappa(\Delta_1, \Delta_1, 0) C_\kappa(\Delta_2, \Delta_2, 0) C_\kappa(\Delta_3, \Delta_3, 0)}}.$$

Theorem A.-Cai-Sun-Wu '24

$$R_\kappa^{\text{nest}}(\widehat{\alpha}_1, \widehat{\alpha}_2, \widehat{\alpha}_3) = C_{\sqrt{\kappa}/2}^{\text{ImDOZZ}}(\widehat{\alpha}_1, \widehat{\alpha}_2, \widehat{\alpha}_3).$$

Physics derivation by [Ikhlef-Jacobsen-Saleur '15], who also gave strong empirical evidence via lattice model approximation.

Discrete interpretation of R_κ^{nest}:
In $O(n)$ loop model, give different weight n_i to loops separating z_i from other two points.

1. Imaginary DOZZ formula

The formula for $C_b^{\mathrm{ImDOZZ}}(\widehat{\alpha}_1, \widehat{\alpha}_2, \widehat{\alpha}_3)$ is

$$A\Upsilon_b(2b - b^{-1} + \sum \widehat{\alpha}_i) \prod_{i=1}^{3} \frac{\Upsilon_b(\widehat{\alpha}_1 + \widehat{\alpha}_2 + \widehat{\alpha}_3 - 2\widehat{\alpha}_i + b)}{\sqrt{\Upsilon_b(2\widehat{\alpha}_i + b)\Upsilon_b(2\widehat{\alpha}_i + 2b - b^{-1})}},$$

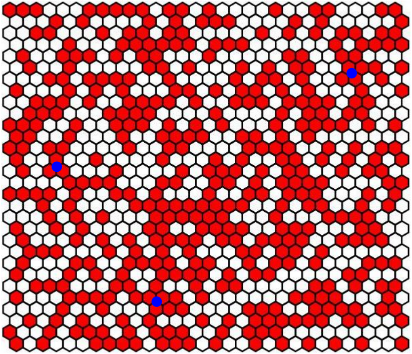
where Υ_b is Zamolodchikov's special holomorphic function defined on $\mathbb{C}$ such that for $Q = b + b^{-1}$ and $0 < \mathrm{Re} z < Q$,

$$\log \Upsilon_b(z) = \int_0^\infty \left((\frac{Q}{2} - z)^2 e^{-t} - \frac{\sinh^2(\frac{Q}{2} - z)}{\sinh(bt)\sinh(b^{-1}t)} \right) \frac{dt}{t}.$$

The **imaginary DOZZ formula** was introduced in physics by [Schomerus '03], [Zamolodchikov '05], [Kostov-Petkova '07] as a structure constant for CFT with central charge $1 - 6(\frac{\sqrt{\kappa}}{2} - \frac{2}{\sqrt{\kappa}})^2$.

Natural extension of the structure constants for minimal model CFTs to continuously varying parameters.

2. Critical percolation connectivities



Critical site percolation on triangular lattice with mesh size δ.

$P_n^\delta(z_1, \ldots, z_n)$ = probability n points lie in the same cluster.

There exists limit [Camia '23]:

$$P_n(z_1, \ldots, z_n) := \lim_{\delta \to 0} \pi_1(\delta)^{-n} P_n^\delta(z_1, \ldots, z_n),$$

where $\pi_1(\delta) = \mathbb{P}[0 \leftrightarrow \partial B_1(0)]$.

2. Three-point connectivity constant

$$P_n(z_1, \ldots, z_n) := \lim_{\delta \to 0} \pi_1(\delta)^{-n} P_n^\delta(z_1, \ldots, z_n).$$

This limit is conformally covariant [Camia '23].

Three-point connectivity constant

$$\frac{P_3(z_1, z_2, z_3)}{\sqrt{P_2(z_1, z_2) P_2(z_2, z_3) P_2(z_3, z_1)}}.$$

- Does not depend on z_1, z_2, z_3.
- Conjecturally universal:
  not dependent on microscopic properties
  (choice of lattice, definition of "lie in the same cluster").

(choice of renormalization factor $\pi_1(\delta)$ disappears in the ratio.)

2. Delfino-Viti conjecture

Conjecture coming from CFT: with $b = \frac{2}{\sqrt{6}}$,

$$\frac{P_3(z_1, z_2, z_3)}{\sqrt{P_2(z_1, z_2)P_2(z_2, z_3)P_2(z_3, z_1)}} =$$

$$\sqrt{2}C_b^{\mathrm{ImDOZZ}}(\frac{1}{4b} - \frac{b}{2}, \frac{1}{4b} - \frac{b}{2}, \frac{1}{4b} - \frac{b}{2}) \approx 1.022.$$

Theorem — A.-Cai-Sun-Wu '24

Above conjecture for the three-point connectivity constant holds.

Baojun Wu's talk on Friday: proof of this result!

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- **Gaussian free field and Gaussian multiplicative chaos.**
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Gaussian free field on the unit disk $\mathbb{D}$

Dirichlet inner product $\langle f, g \rangle_\nabla := \frac{1}{2\pi} \int_{\mathbb{D}} \nabla f(z) \cdot \nabla g(z) \, dz$.

Consider the space of smooth functions f on $\overline{\mathbb{D}}$ with $\langle f, f \rangle_\nabla < \infty$ and mean zero on $\partial \mathbb{D}$.

Let $H(\mathbb{D})$ be its closure with respect to $\langle \cdot, \cdot \rangle_\nabla$.

Let $f_1, f_2, \ldots$ be an orthonormal basis of $H(\mathbb{D})$.

Sample $a_1, a_2, \ldots$ i.i.d. $N(0,1)$ and set

$$h = \sum_i a_i f_i.$$

This limit a.s. exists in the space of distributions
(more strongly, $h \in H^{-\varepsilon}_{\mathrm{loc}}(\mathbb{D})$ for any $\varepsilon > 0$).

h is the **(free boundary) GFF with mean zero on $\partial \mathbb{D}$**.

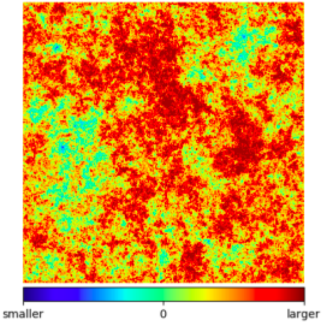
Gaussian free field

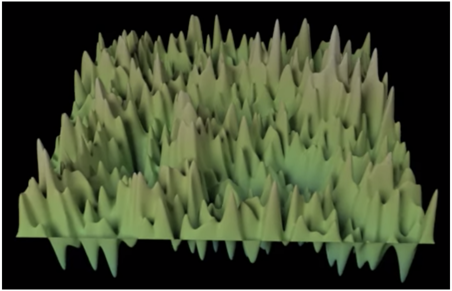
Mean zero Gaussian field h on $\mathbb{D}$ with covariance

$$\mathbb{E}[h(z)h(w)] = -\log|z-w| - \log|1-z\overline{w}|.$$

h is a **distribution** or **generalized function**:

- $h(z)$ is not well-defined;
- $\int_{\mathbb{C}} h(z)\rho(dz)$ is defined when ρ is sufficiently regular, e.g., $\rho(dz) = f(z)dz$ for smooth compactly supported f.





(Simulations by Minjae Park, Henry Jackson.)

Gaussian free field

Mean zero Gaussian field h on $\mathbb{D}$ with covariance

$$\mathbb{E}[h(z)h(w)] = -\log|z-w| - \log|1 - z\overline{w}|.$$

h is a **distribution** or **generalized function**:

- $h(z)$ is not well-defined;
- Let $h_\varepsilon(z)$ be the average of h on radius-ε circle around z.
- h is normalized so that $h_1(0) = 0$.

Do a calculation on the blackboard:

$$\operatorname{Var} h_\varepsilon(z) = -\log\varepsilon + O(1) \qquad \text{for } z \in \mathbb{D}.$$

$$\varepsilon^{\gamma^2/2} e^{\gamma h_\varepsilon(z)} \text{ small with high probability,}$$

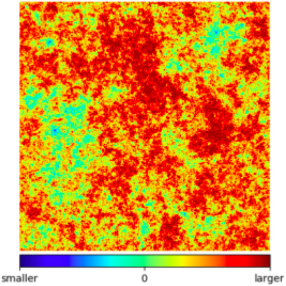
$$\text{but } \varepsilon^{\gamma^2/2}\mathbb{E}[e^{\gamma h_\varepsilon(z)}] \text{ is constant order.}$$

Gaussian multiplicative chaos

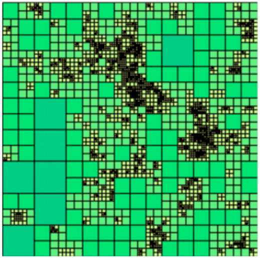
Let h be a Gaussian free field on $\mathbb{D}$.
Let $h_\varepsilon(z)$ be the average of h on the radius-ε circle around z.
Let $\gamma \in (0,2)$. Can define a measure

$$A_h^\gamma(dz) := \lim_{\varepsilon \to 0} \varepsilon^{\gamma^2/2} e^{\gamma h_\varepsilon(z)}\, dz.$$

This is an example of **Gaussian multiplicative chaos**.
Kahane '85, Robert-Vargas '08, Duplantier-Sheffield '08



Gaussian free field h
Images by Minjae Park.



Discretization of A_h^γ,
squares have comparable A_h^γ-mass.

Gaussian multiplicative chaos

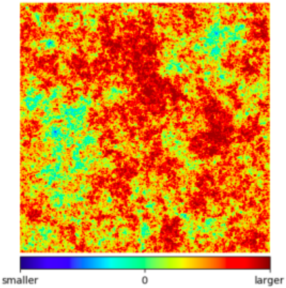
Let $\gamma \in (0,2)$. Can define a measure

$$A_h^\gamma(dz) := \lim_{\varepsilon \to 0} \varepsilon^{\gamma^2/2} e^{\gamma h_\varepsilon(z)}\, dz.$$

Measure A_h^γ is supported on the set of γ-**thick points**:

$$T_h^\gamma = \{z \in \mathbb{D} \,:\, \lim_{\varepsilon \to 0} h_\varepsilon(z)/\log(1/\varepsilon) = \gamma\}.$$

Hausdorff dimension of T_h^γ is $2-\gamma$.



Gaussian free field h

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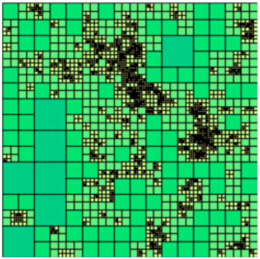

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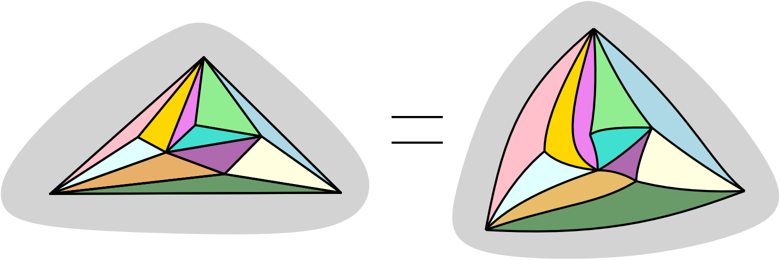
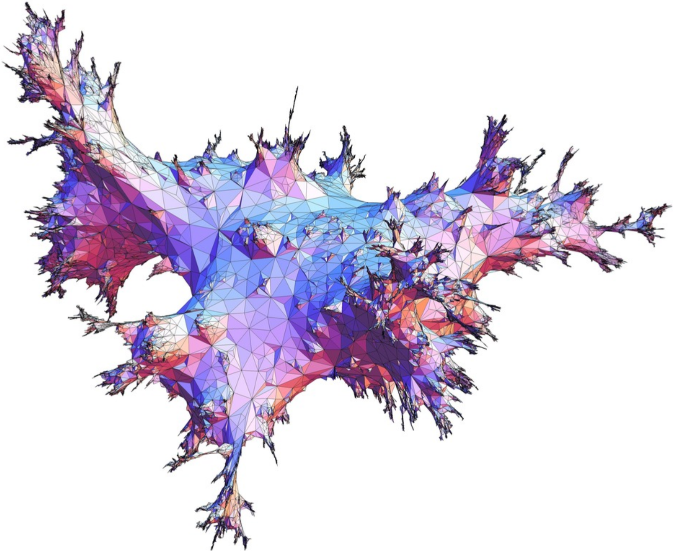
What is a planar map?



Embedding of planar graph into Riemann sphere $\mathbb{C} \cup \{\infty\}$, modulo homeomorphisms of $\mathbb{C} \cup \{\infty\}$.

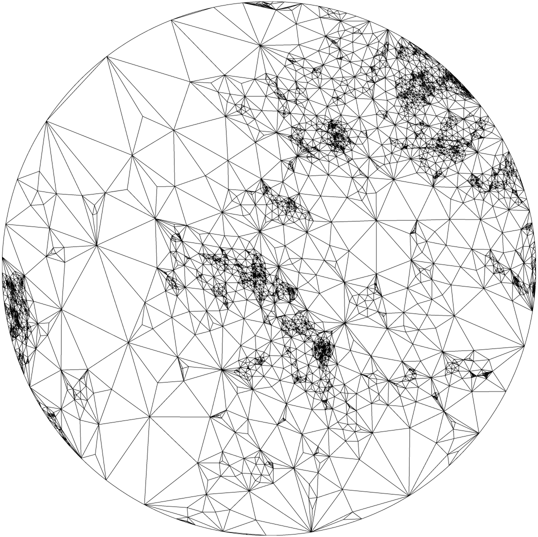
Uniform random planar map is a uniform sample from the set of planar maps with n faces.

Random planar map simulation



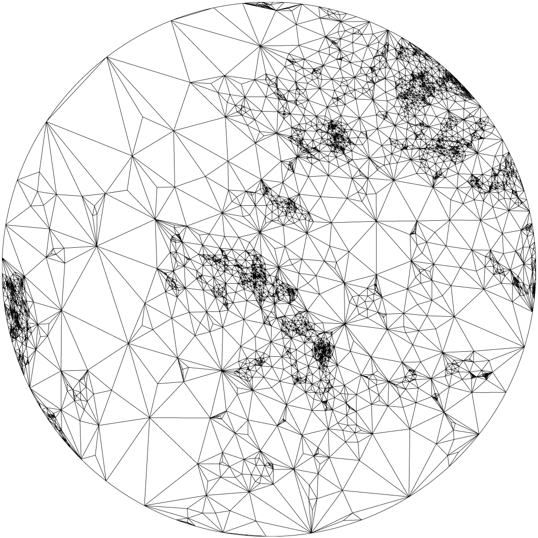
Random planar map, spring embedding in $\mathbb{R}^3$ (Thomas Budzinski).

Random planar map simulation

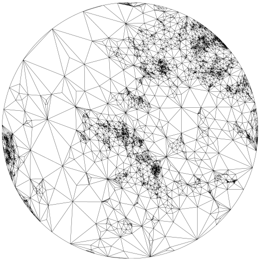


Random planar map, harmonic embedding in $\mathbb{D}$ (Jason Miller).

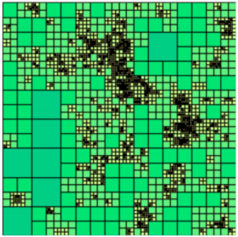
Random planar map simulation



Random planar map, harmonic embedding in $\mathbb{D}$ (Jason Miller).
Scaling limit of this random geometry is **Liouville quantum gravity**.



n-**face random planar map**
Conformally embedded in $\mathbb{D}$.

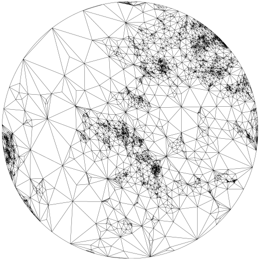


Liouville quantum gravity

Discrete area measure A_n, boundary measure L_n, metric D_n.

General conjecture:
(A_n, L_n, D_n) converges to a continuum triple $(A_h^\gamma, L_h^\gamma, D_h^\gamma)$ called **Liouville quantum gravity** defined via Gaussian free field.

Random planar maps → Liouville quantum gravity



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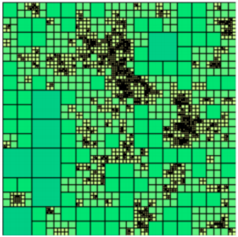

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Proved for important special cases:
Gwynne-Miller-Sheffield '17 (mated-CRT map, harmonic embedding),
Holden-Sun '19 (uniform RPM, Cardy embedding),
Bertacco-Gwynne-Sheffield '23 (mated-CRT map, Smith embedding).

Liouville quantum gravity parameter γ

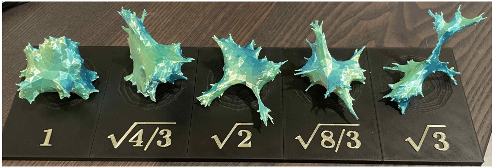
"Larger $\gamma \implies$ rougher surface".

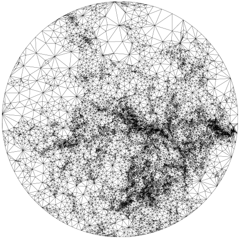
$\gamma = \sqrt{2}$: Tree-decorated random planar map.

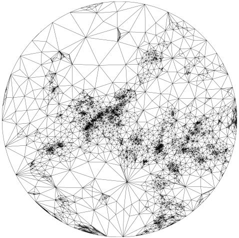
$\gamma = \sqrt{8/3}$: Uniform random planar map.

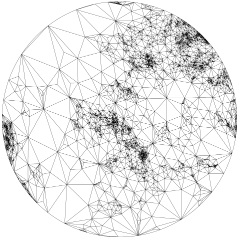
$\gamma = \sqrt{3}$: Ising-decorated random planar map.

$\gamma = 2$: GFF-decorated random planar map









Two perspectives on Liouville quantum gravity

Liouville conformal field theory (LCFT)

David, Guillarmou, Kupiainen, Rhodes, Vargas, ...

- 2D quantum field theory with conformal symmetries.
- Object of interest: Correlation function.

Random surface perspective

Sheffield, Duplantier-Miller-Sheffield, ...

- Describe scaling limit of random planar maps.
- Object of interest: quantum surface
  ("= measure space with conformal structure").

These two perspectives describe the same random geometry!
Lecture II: Present both perspectives, explain equivalence.

Liouville conformal field theory

A **quantum field theory** is a collection of numbers called **correlation functions**, that arise as expectations of a random field.

A **conformal field theory** is a QFT with conformal symmetries.

Liouville conformal field theory was introduced by Polyakov '81 in the context of bosonic string theory.

One of the most fundamental 2D CFTs.

Mathematically constructed by

- David-Kupiainen-Rhodes-Vargas '14 (sphere)
- Huang-Rhodes-Vargas '15 (disk)
- Guillarmou-Rhodes-Vargas '16 (closed surfaces)
- Remy '17 (annulus)

Liouville CFT: the Liouville field on $\mathbb{D}$

Let $\gamma \in (0,2)$ and $Q = \frac{\gamma}{2} + \frac{2}{\gamma}$.
Let $P_{\mathbb{D}}$ be the law of the GFF on $\mathbb{D}$ with average zero on $\partial\mathbb{D}$.

Liouville field on $\mathbb{D}$ — Huang-Rhodes-Vargas '15

Sample $(h, \mathbf{c})$ from $P_{\mathbb{D}} \times [e^{-Qc}dc]$ on $H^{-1}(\mathbb{D}) \times \mathbb{R}$.
The Liouville field is $\phi = h + \mathbf{c}$.
Let $\mathrm{LF}_{\mathbb{D}}$ be the measure on $H^{-1}(\mathbb{D})$ describing the law of ϕ.

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Formally, Liouville field with insertion of size $\alpha \in \mathbb{R}$ at $z \in \mathbb{D}$ is $\mathrm{LF}_{\mathbb{D}}^{(\alpha,z)} = e^{\alpha\phi(z)} \mathrm{LF}_{\mathbb{D}}(d\phi)$.

Can rigorously define as $\mathrm{LF}_{\mathbb{D}}^{(\alpha,z)} := \lim_{\varepsilon \to 0} \varepsilon^{\alpha^2/2} e^{\alpha\phi_\varepsilon(z)} \mathrm{LF}_{\mathbb{D}}(d\phi)$.

Near z, field looks like GFF $- \alpha \log|\cdot - z|$.

LCFT disk one-point correlation function for $\mu, \mu_B \geq 0$:

$$\langle e^{\alpha\phi(0)} \rangle_{\mathbb{D}} := \int e^{-\mu A_\phi(\mathbb{D}) - \mu_B L_\phi(\partial\mathbb{D})} \mathrm{LF}_{\mathbb{D}}^{(\alpha,0)}(d\phi).$$

Encodes law of $(A_\phi(\mathbb{D}), L_\phi(\mathbb{D}))$ for $\phi \sim \mathrm{LF}_{\mathbb{D}}^{(\alpha,0)}$.

Correlation functions of Liouville CFT

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Can define Liouville field, correlation functions for general surfaces, e.g. Riemann sphere $\widehat{\mathbb{C}}$:

$$\langle \prod_{i=1}^{3} e^{\alpha_i \phi(z_i)} \rangle_{\widehat{\mathbb{C}}} := \int e^{-\mu A_\phi(\widehat{\mathbb{C}})} \mathrm{LF}_{\widehat{\mathbb{C}}}^{(\alpha_1,z_1),(\alpha_2,z_2),(\alpha_3,z_3)}(d\phi).$$

Correlation functions of Liouville CFT

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"Solving LCFT" = computing all correlation functions.

Properties of LCFT correlation functions

Let $\gamma \in (0,2)$ and $Q = \frac{\gamma}{2} + \frac{2}{\gamma}$. Following are necessary properties for conformal field theory.

Diffeomorphism invariance — Huang-Rhodes-Vargas '18

Suppose f is a diffeomorphism from (D, g) to (D', g'). Then

$$\langle F(\phi) \rangle_{D',g',\mu,\mu_\partial} = \langle F(f \circ \phi) \rangle_{D,g,\mu,\mu_\partial}.$$

Weyl anomaly — Huang-Rhodes-Vargas '18

Suppose $g' = e^{2\sigma} g$. Then

$$\frac{\langle F(\phi + Q\sigma) \rangle_{D,g',\mu,\mu_\partial}}{\langle F(\phi) \rangle_{D,g,\mu,\mu_\partial}} = \left(Z^{\mathrm{FF}}_{g'} / Z^{\mathrm{FF}}_{g} \right)^{1+6Q^2},$$

where $Z^{\mathrm{FF}}_g = (\det{}' \Delta_g)^{-1/2}$ and $\det{}' \Delta_g$ is the zeta-regularized determinant of Laplacian.

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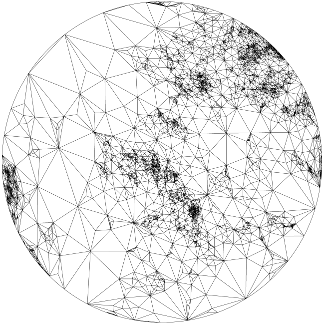
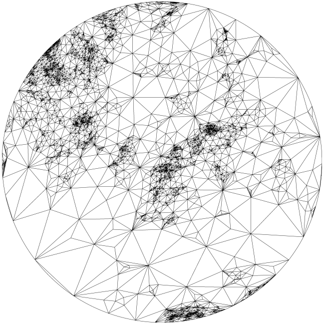
Random surface perspective of LQG

Conformal embedding of a random planar map is not unique!

e.g., can apply conformal automorphism:

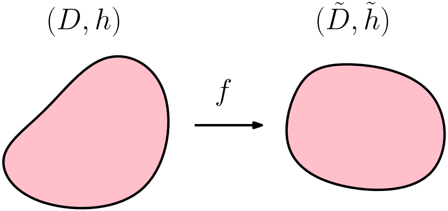




In definition of random surface, we will “mod out by choice of conformal embedding”.

Random surface perspective of LQG

Let $f : D \to \tilde{D}$ be a conformal map. Let $Q = \frac{\gamma}{2} + \frac{2}{\gamma}$.



$$h = \tilde{h} \circ f + Q \log |f'|$$

Define $(D, h) \sim_\gamma (\widetilde{D}, \widetilde{h})$ if $h = \widetilde{h} \circ f + Q \log |f'|$.
All members of equivalence class have same LQG measures:

$$f_* A_h^\gamma = A_{\widetilde{h}}^\gamma, \qquad f_* L_h^\gamma = L_{\widetilde{h}}^\gamma.$$

A **quantum surface** is an equivalence class $(D, h)/\sim_\gamma$.
It is a **measure space with conformal structure**.
(D, h) is called an **embedding** of the quantum surface.

Special quantum surfaces: quantum sphere (motivation)

Fix $\gamma = \sqrt{8/3}$.

Consider a kind of planar map (triangulations, quadrangulations, ...), let S_n = set of maps with n vertices.

$$|S_n| = \text{const} \cdot e^{\beta n} \cdot n^{-7/2}(1 + o_n(1)).$$

Here const and β are nonuniversal, but $-7/2$ is universal.

Two ways to remove nonuniversal terms:

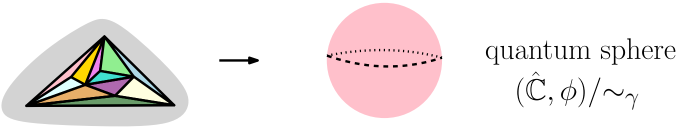
- Fix n. In the scaling limit get **unit area quantum sphere**.
- Consider $\bigcup_n S_n$ but weight each map by $e^{-\beta n}$.
  Scaling limit gives **(free area) quantum sphere**.

Special quantum surfaces: quantum sphere

Fix $\gamma \in (0,2)$.

Unit area quantum sphere is a quantum surface.

Constructed via GFF, motivated by scaling limit of RPM.



$\mathrm{QS}(a)^\# =$ law of area a quantum sphere.

Non-probability measure $\mathrm{QS}(a) = \text{const} \times a^{-\frac{4}{\gamma^2}-2}\mathrm{QS}(a)^\#$.

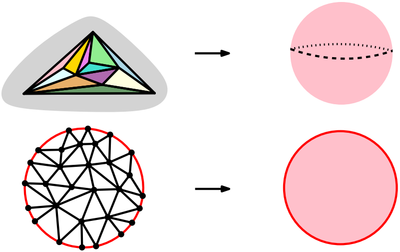
Quantum sphere with no area restriction:

$$\mathrm{QS} := \int_0^\infty \mathrm{QS}(a)\, da.$$

Note: QS is an infinite measure on the space of quantum surfaces!

Important quantum surfaces

Fix $\gamma \in (0, 2)$.



quantum sphere
$(\hat{\mathbb{C}}, \phi)/{\sim_\gamma}$

quantum disk
$(\mathbb{D}, \phi)/{\sim_\gamma}$

Laws are infinite measures, which we denote by QS and QD.

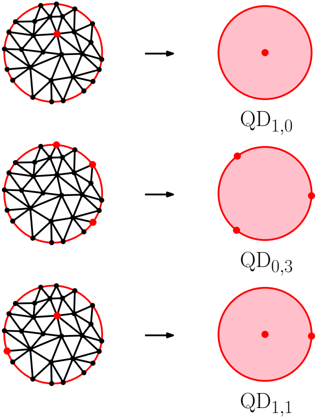
$\mathrm{QD}(\ell)$ = law of quantum disks with boundary length ℓ. Then

$$\mathrm{QD} = \int_0^\infty \mathrm{QD}(\ell)\, d\ell, \qquad |\mathrm{QD}(\ell)| \propto \ell^{-\frac{4}{\gamma^2}-2}.$$

$|\mathrm{QD}(\ell)|$ = partition function of boundary length ℓ quantum disks.

Quantum disk with marked points

Fix $\gamma \in (0,2)$ and $m, n \geq 0$.
Sample $\mathcal{D}$ from weighted measure $A(\mathcal{D})^m L(\mathcal{D})^n \mathrm{QD}(d\mathcal{D})$,
independently sample m bulk points and n boundary points.
Let $\mathrm{QD}_{m,n}$ be the law of the $(m+n)$-pointed quantum surface.



e.g., $\qquad |\mathrm{QD}_{0,3}(\ell)| = \ell^3 |\mathrm{QD}(\ell)|.$

Equivalence of perspectives: QS_3

A sample from QS_3 is a quantum surface with three marked points.

Infinitely many choices of embedding in $\widehat{\mathbb{C}}$, but can uniquely specify embedding by fixing the location of the three points in $\widehat{\mathbb{C}}$.

Theorem — Aru-Huang-Sun '17

Let $z_1, z_2, z_3 \in \mathbb{C}$ be distinct.
Sample from QS_3 and let $(\widehat{\mathbb{C}}, \phi, z_1, z_2, z_3)$ be the embedding which places the points at z_1, z_2, z_3.
Then the law of ϕ is $C_{z_1,z_2,z_3} \mathrm{LF}_{\widehat{\mathbb{C}}}^{(\gamma,z_1),(\gamma,z_2),(\gamma,z_3)}$ for some constant C_{z_1,z_2,z_3}.

Equivalence of perspectives: $\mathrm{QD}_{0,3}$ and $\mathrm{QD}_{1,1}$

Theorem (Cerclé '21)

Let $x_1, x_2, x_3 \in \partial\mathbb{D}$ be distinct.
Sample from $\mathrm{QD}_{0,3}$ and embed it as $(\mathbb{D}, \phi, x_1, x_2, x_3)$.
Then the law of ϕ is $C_{x_1,x_2,x_3}\mathrm{LF}_{\mathbb{D}}^{(\gamma,x_1),(\gamma,x_2),(\gamma,x_3)}$ for some constant C_{x_1,x_2,x_3}.

Theorem (A.-Remy-Sun '21)

Let $z \in \mathbb{D}$ and $x \in \partial\mathbb{D}$.
Sample from $\mathrm{QD}_{1,1}$ and embed it as $(\mathbb{D}, \phi, z, x)$.
Then the law of ϕ is $C_{z,x}\mathrm{LF}_{\mathbb{D}}^{(\gamma,z),(\gamma,x)}$ for some constant $C_{z,x} \in (0, \infty)$.

Equivalence of perspectives: applications

Each perspective has its own tools.
Using the equivalence of perspectives, can access both sets of tools!

- Values of Liouville CFT correlation functions are crucial inputs in computing observables of SLE and LQG+SLE.
  [A.-Holden-Sun '21], [A.-Sun '21], [A.-Remy-Sun '22].

- Conversely, random surface methods (e.g. SLE) can be used to solve for Liouville CFT correlation functions.
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Next lecture: details on the following tools/solvability.

- Liouville CFT corr functions can be solved via BPZ and OPE.
- LQG+SLE coupling, inputs from random planar maps.