The Elastic Manifold

Gérard BEN AROUS

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1. Disordered Elastic Media, or The Elastic manifold

1. The Model

"Many seemingly different systems ranging from magnets to superconductors, with extremely different microscopic physics share the same essential ingredients, and can be described under the unifying concept of disordered elastic media. In all these systems an internal elastic structure, such as an interface between regions of opposite magnetizations in the magnetic systems, is subjected to the effects of disorder existing in the material... What properties result from the competition between elasticity and disorder is an extremely complicated problem which constitutes the essence of the physics of disordered elastic media."
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- One typically start with a "confining" potential like the harmonic potential, and then add disorder, in the form of a random potential depending on the position x and on the value of the field $u(x)$

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- We can of course also add an external field term (say $\sqrt{N}h\sum_x < u(x), e >$)which is trying to "depin" the elastic manifold in one specific direction (here the unit vector e).

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- V_N is a centered smooth isotropic Gaussian field with covariance

$$\mathbb{E}[V_N(x,u)V_N(y,v)] = \delta_{x,y} NB(\frac{1}{N}||u-v||^2) \tag{5}$$

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where ν is a finite positive measure.
- We assume here that B is 4 times differentiable, which ensures that V_N is C^2, and that $B^{(i)}(0) \neq 0$ for $i = 0, 1, 2$ to avoid degeneracies.

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- The next question is to understand the behavior of the Gibbs measure defined by this Hamiltonian at positive temperature, and the structure of the RSB phase (replica symmetry breaking). The first step is naturally to understand the quenched free energy and prove a Parisi formula. This is done in three works with Pax Kivimae, which I will try to summarize today.

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- This problem was studied massively in Physics, back to Fischer (1986), Mezard-Parisi (1991 and 1992) and a long literature in Physics.
- More recently by Le Doussal-Mueller-Wiese 2007, and Fyodorov-Le Doussal (2020) for a result on complexity that motivated this work.

2. A summary of our results

1. The complexity of the Elastic Manifold

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5. These results confirm fully the recent work by Fyodorov and Le Doussal (2020).

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2. We prove a Parisi formula for the quenched free energy, using the well established tools as developed by Guerra, Talagrand, Panchenko (see for instance Panchenko's book), as well the specific tools used for multi-species Spin Glasses by Panchenko (synchronization). In fact we do this in a wider class of models (in our first paper)

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2. In particular we study its consequence on Replica Symmetry Breaking at low temperature

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1. But we can simplify this formula again and go from a "bad" Mezard-Parisi formula to a good one, and then identify the optimizer and deduce consequences on the Gibbs measure (in our second paper).
2. In particular we study its consequence on Replica Symmetry Breaking at low temperature

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5. And of course we do not study the other (non mean-field) limits.

Spin Glasses

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$$H_{N,p}(x) = \frac{1}{N^{\frac{p-1}{2}}} \sum_{i_1 \dots i_p = 1}^{N} J_{i_1 \dots i_p} x_{i_1} \dots x_{i_p} \tag{11}$$

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- Here $\xi(u) = \sum_{p=2}^{\infty} a_p^2 u^p$ specifies the Spin Glass model
- And $R_N(x, y) = \frac{<x,y>}{N} = \frac{1}{N} \sum_{i=1}^{N} x_i y_i \in [-1, 1]$ is usually called the "overlap" of x and y.

4. The Gibbs measure

- The first question is about statics, or equilibrium, i.e. to understand the behavior of the Gibbs measure on Σ_N as $N \to \infty$:

$$G_{N,\beta}(dx) = \frac{1}{Z_N(\beta)} e^{-\beta H_N(x)} \mu_N(dx) \tag{14}$$

where

- μ_N is the natural uniform measure on Σ_N
- $\beta = \frac{1}{T}$ is the inverse temperature, and
- $Z_N(\beta)$ is the partition function, i.e the normalizing constant to make the Gibbs measure a (random) probability measure on Σ_N

$$Z_N(\beta) = \int_{\Sigma_N} e^{-\beta H_N(x)} \mu_N(dx) \tag{15}$$

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- Can one understand the thermalization time?

Statics: the Parisi approach

6. The free energy and Parisi's formula

- This equilibrium question in fact begins with the understanding of the behavior of the partition function Z_N, or rather its logarithm, i.e. the free energy.

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- The main result (valid both for the Ising spins or spherical spins cases) is that the limiting free energy exists almost surely and is given by the famous Parisi variational formula

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- Note that the formula is not standard. This should be a sup not an inf !

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- And $\hat{q}$ is any point on the right of the support, i.e. such that $\mu([0,\hat{q}]) = f_\mu(\hat{q}) = 1$

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- This is still true at high enough temperature ($T > T_s$): the minimizer in the Parisi formula is simply δ_0. This is called the Replica Symmetric Phase (RS). Two points independently sampled from the Gibbs measure are roughly orthogonal.
- At low temperature things are more interesting ($T < T_s$). If the unique minimizer of the Parisi functional is the sum of $k+1$ atoms, the model is said to be in a k-Replica Symmetry Breaking Phase (k-RSB). If the minimizer contains a continuous part the model is in Full Replica Symmetry Breaking Phase (FRSB).

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The landscape complexity approach

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- Of course note that $R_N(x,y) = \frac{<x.y>}{N} = 1 - \frac{||x-y||^2}{2N}$. So that these covariances are in fact functions of the Euclidean distance (or of the distance on the sphere).

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- From now on, we restrict to the case of spherical models.
- As we have seen their Hamiltonian define smooth random centered Gaussian functions on the sphere Σ_N, with covariance structure given by

$$\mathbb{E}(H_N(x)H_N(y)) = N\xi(R_N(x,y)) \tag{20}$$

- Where

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- The answer in a nutshell: there are exponentially many critical points and minima (for the pure p -spin when $p \geq 3$). The energy landscape is topologically very "rough".
- The method is based on the Kac-Rice formula, which gives a dictionary from these random geometry questions to Random Matrix Theory.

16. The Landscape Complexity picture for the pure p-spin

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- Annealed Estimates: Let $Crit_k(E)$ be the number of critical points of index k, and energy value less than NE, then the following limit exists and cam be computed precisely, using a LDP for the kth eigenvalue of the GOE

$$\lim_{N\to\infty} \frac{1}{N} \log \mathbb{E}[Crit_k(E)] = \Theta_k(E)$$

  1. There is a value $E_k(p)$ such that $\Theta_0(E) > 0$ for $E > -E_k$.
  2. The sequence $-E_0 < -E_1 < -E_2 ...$ is increasing and converges to the threshold energy $-E_\infty$
  3. The function Θ_k is increasing in the interval $(-\infty, -E_\infty)$, and is constant above $-E_\infty$.

- Quenched Estimates: Subag used a second moment method and Kac Rice to prove that the quenched estimates are also valid for energies E close to the ground state $-E_0$.

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- See (Fyodorov 2005, Auffinger-GBA-Cerny 2013, Auffinger-GBA 2013) for results giving the "annealed" complexity i.e. the behavior of $\frac{1}{N} \log \mathbb{E}(Crit_k(E))$, where $Crit_k(E)$ is the number of critical points of index k, and below level E.

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- See Subag 2017-2018 for quenched ones, i.e. the behavior of $\frac{1}{N}\mathbb{E}(\log(Crit_k(E)))$, for very low energy levels E, and also Auffinger-Gold 2019 for quenched result on critical points of higher index.

18. The geometry of the Gibbs measure at low temperature

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- These local minima are the deepest ones for the pure p-spin model, and slightly higher ones for the mixed models (their center change with temperature, which induces temperature chaos)

Above the static transition: the shattering phase

20. The geometry of the Gibbs measure above the static transition

- In a more recent work with Aukosh Jagannath, we now try to study the behavior of the spherical p-spin model (its free energy, and possibly the Gibbs measure) at *higher* temperatures, i.e in a regime where $T > T_s$, thus where the system is in the Replica Symmetric phase.

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- But, as we will see, this is detected by the more precise topological complexity approach.

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- It was later studied in great depth for many important problems related to sparse mean-field models of spin glasses (Dembo-Montanari-Sun 2013), and central questions from Theoretical Computer Science and Combinatorics, such as random constraint satisfaction (Mezard-Parisi-Zecchina 2002, Krzakala-Montanari-Ricci Tersenghi 2007, Ding-Sly-Sun 2015) and combinatorial optimization problems (Achlioptas-Coja Oghlan 2008, Sly-Zhang 2016).

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- In this work, we return to the dense case of spherical p-spin models following the early and fundamental paper by Barrat, Burioni, and Mézard [12].

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- The REM is a model for Ising spins $\Sigma_N = \{-1, 1\}^N$, where the $H_N(\sigma)$ are just i.i.d $N(0, N)$ variables. So that

$$Z_N(\beta) = \sum_{\sigma \in \{-1,1\}^N} e^{-\beta H_N(\sigma)}$$

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- This β_d is also the onset of the dynamical phase transition, and of interesting aging regimes.

25. The shattering phase for the p-spin model

- Consider, for $A \subset \Sigma_N$, the free energy on A

$$F_N(\beta, A) = \frac{1}{N} \log \int_A e^{-\beta H_N(x)} d\mu_N(x) \tag{22}$$

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- For a fixed $T > 0$, $E \in \mathbb{R}$, $r \geq 0$, and $0 < q < 1$, we say the free energy landscape is (E, q, r)-*shattered* at temperature T iff
- There exist $c, c' > 0$ and sequences $\epsilon_N, \eta_N, \delta_N \to 0$, such that, with probability tending to 1, the following occurs:

26. The shattering phase for the p-spin model

There is a sequence of sets $A_N \subset Crit_N([-E - \epsilon_N, -E + \epsilon_N])$, such that for $\beta = T^{-1}$,

1. (positive complexity) $\frac{1}{N} \log |A_N| \geq c$,
2. (separation) for all distinct $x, y \in A$, we have that $B(x, q, \eta_N) \cap B(y, q, \eta_N) = \emptyset$ and that $R(x, y) < r$,
3. (sub-dominance) and for each $x \in A$, the band $B(x, q, \eta_N)$ is c'-subdominant,

$$F_N(\beta) - F_N(B(x, q, \eta_N); \beta) > c' > 0.$$

   or equivalently

$$G_{N,\beta}(B(x, q, \eta_N)) \leq e^{-c'N}$$

4. (free energy equivalence) Furthermore, we have that

$$F_N(\beta) - F_N(\cup_{x \in A_N} B(x, q, \eta_N), \beta) \to 0$$

   in probability.

27. Existence of the shattering phase

- Theorem (Jagannath, GBA CPAM 2023)
  For $p \geq 4$, there exists a $T_0 \in (T_s, T_{sh}]$ where $T_{sh} = \sqrt{\frac{p(p-2)^{p-2}}{(p-1)^{p-1}}}$, such that the free energy landscape is $(E(\beta), q(\beta), r)$ shattered for all $T_s < T < T_0$.

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Slow dynamics at high temperature

28. Langevin Dynamics

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3. Back to the Elastic Manifold: The quenched free energy and Replica Symmetry Breaking

1. A first Mezard-Parisi formula for the free energy

- The first result is the existence of the quenched limiting free energy $f(\beta)$: As $N \to \infty$, the limit $f(\beta) = \frac{1}{NL^d} \log Z_{N,\beta}$ exists a.s and in expectation

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- The functional $P_\beta(q,\zeta)$ is defined by

$$P_\beta(q,\zeta) = \frac{1}{2}(\log(\frac{2\pi}{\beta}) + \beta\frac{h^2}{\mu} - \beta\mu q$$

$$+\Lambda(\beta(q-q_*)) + \int_0^{q*} \beta K(\beta\delta(u))du - 2\beta^2 \int_0^q \zeta([0,u])B'(2(q-u))du)$$

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- Λ is in fact the Legendre transform of the strictly concave function $\frac{1}{L^d} \log\det(K(u)I - t\Delta)$.

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- For any bounded continuous function f

$$\lim_{N\to\infty} E(f((u(x), u'(x))_N) = \int f(r + \frac{h^2}{\mu^2})\zeta_\beta(dr)$$

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$$4B''(2\frac{R_1(\mu)}{\beta}) \leq \frac{1}{R_2(\mu)}$$

- The Larkin mass is decreasing with β, and $L(\infty)$ is indeed the limit of $L(\beta)$ as $\beta \to \infty$.

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- If $\mu < \mu_L(\beta)$, then the model is RSB at (β, μ) for β large enough.
- If $4B''(\frac{2R_1(\mu)}{\beta})R_2(\mu) > 1$ the model is RSB.

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- This is already a problem in the one-site case at fixed temperature (subject to a slight non-degeneracy condition) the Larkin mass $\mu_L(\infty)$ detects the first, but perhaps not the last, shift between RS-RSB phases!!

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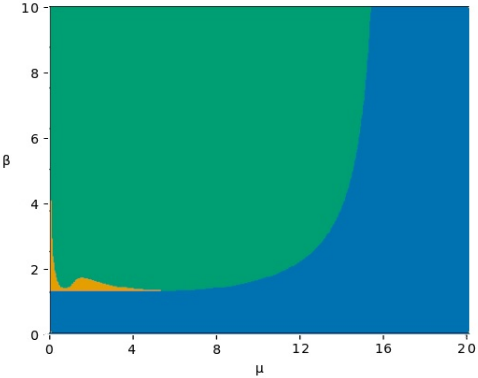
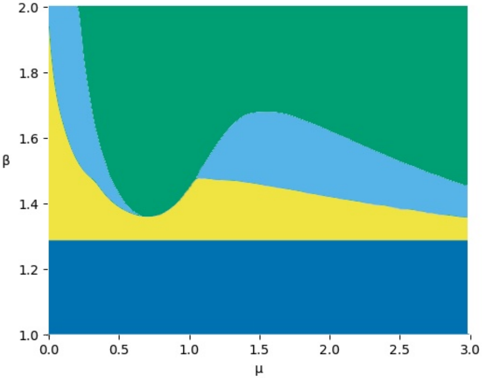
12. RS/RSB Phase Diagram



- Phase Diagram in the case $B(x) = e^{-x} + e^{-8x}$ and $|L^d| = 1$.
- Dark Blue is the region above the Larkin mass, which is RS. Green is RSB by the local condition. Orange is neither.

13. Zoomed in: RS/RSB Phase Diagram



- Phase Diagram in the case $B(x) = e^{-x} + e^{-8x}$ and $|L^d| = 1$.
- Green and Light Blue regions are RSB. Yellow and Dark Blue regions are RS.

4. Stating the results for the topological complexity

1. Annealed Complexity

- Let $\mathcal{N}_{tot}$ the (random) number of all critical points of the Elastic Manifold Hamiltonian $H(u)$, then
- Theorem (BA-Bourgade, McKenna, Arxiv 2021, CPAM 2024) The annealed total complexity is given by

$$\lim_{N\to\infty} \frac{1}{NL^d} \log \mathbb{E}[\mathcal{N}_{tot}] = \Sigma(\mu_0, t_0, b) \tag{23}$$

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- Similarly let $\mathcal{N}_m$ the (random) number of all local minima of the Elastic Manifold Hamiltonian

$$\lim_{N\to\infty} \frac{1}{NL^d} \log \mathbb{E}[\mathcal{N}_{min}] = \Sigma_{min}(\mu_0, t_0, b) \tag{24}$$

where the functions Σ and Σ_{min} are explicit and $b = 4B''(0)$.

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- Consider the simple (and deterministic) real-symmetric matrix of size L^d given by

$$D(\mu_0, t_0) = \mu_0 Id - t_0 \Delta \tag{25}$$

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- Consider the simple (and deterministic) real-symmetric matrix of size L^d given by

$$D(\mu_0, t_0) = \mu_0 Id - t_0 \Delta \tag{25}$$

and its spectral measure

$$\mu(t_0, \mu_0) = \frac{1}{L^d} \sum_{i=1}^{L^d} \delta_{\lambda_i} \tag{26}$$

and finally denote by σ_b the semi-circle measure of radius $2\sqrt{b} > 0$

3. The complexity functions Σ and Σ_{min}

- Then we have the variational formulae

$$\Sigma(\mu_0, t_0, b) = -\frac{1}{L^d} \log \det D(\mu_0, t_0) + \tag{27}$$

$$\sup_{u \in \mathbb{R}} \left(\int \log |\lambda - u| (\sigma_b \boxplus \mu(t_0, \mu_0))(d\lambda) - \frac{u^2}{2b} \right) \tag{28}$$

and

$$\Sigma_{min}(\mu_0, t_0, b) = -\frac{1}{L^d} \log \det D(\mu_0, t_0) + \tag{29}$$

$$\sup_{u \leq \ell} \left(\int \log |\lambda - u| (\sigma_b \boxplus \mu(t_0, \mu_0))(d\lambda) - \frac{u^2}{2b} \right) \tag{30}$$

where $\ell = \ell(t_0, \mu_0)$ is the left end of the support of the free convolution $\sigma_b \boxplus \mu(t_0, \mu_0)$.

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- For t_0 and b given, define the "Larkin mass" as the unique solution $\mu_c = \mu_c(t_0, b)$ of

$$\int \frac{1}{(\mu_c + \lambda)^2} \hat{\mu}_{-t_0\Delta}(d\lambda) = \frac{1}{b} \tag{31}$$

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- Then, when $\mu \geq \mu_c$, both the total and the minima complexities $\Sigma(\mu_0, t_0, b)$ and $\Sigma_{min}(\mu_0, t_0, b)$ vanish! i.e. " a large enough mass kills the exponential complexity of the Landscape " !!

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- We could also phrase this by saying that the complexities vanish when the noise level $b = 4B''(0)$ is lower than the critical noise level

$$b_c = b_c(t_0, \mu_0) = \left(\int \frac{1}{(\mu_0 + \lambda)^2} \hat{\mu}_{-t_0\Delta}(d\lambda)\right)^{-1} \tag{32}$$

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6. Topological complexity below the Larkin mass

- Moreover, below the Larkin mass, both annealed complexities are positive, and explicit (I'll spare you the gory details)
- Indeed, the supremum in the formula giving the total complexity is achieved at an explicit $v \in \mathbb{R}$
- The supremum in the formula giving the minima complexity is achieved at $\ell(t_0, \mu_0)$, i.e. the left end of the support of the free convolution $\sigma_b \boxplus \mu(t_0, \mu_0)$

7. The phase transition at the Larkin mass or at the critical noise level

- We express it here in terms of the noise level $b = 4B''(0)$.

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- We express it here in terms of the noise level $b = 4B''(0)$.
- When b approaches the critical level b_c from above, the total annealed complexity vanishes quadratically, and the minima annealed complexity vanishes cubically as a function of the noise level

$$\Sigma(\mu_0, t_0, b) = c_{tot}(b - b_c)^2 + O((b - b_c)^3) \tag{33}$$

$$\Sigma_{min}(\mu_0, t_0, b) = c_{min}(b - b_c)^3 + O((b - b_c)^4) \tag{34}$$

8. The proofs in a nutshell

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4. Apply Laplace's formula, and get a (very heavy) variational formula on $\mathbb{R}^{L^d}$

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2. Recognize this variational problem as related to the problem in $d = 0$, i.e,. for one point. This problem is a "spin glass" type model: the soft spin in an anisotropic random potential!!
3. Use our understanding of this spin glass problem, as mentioned below, to deduce the results about the topological complexity and the topological transition for the Elastic manifold.

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- This has been done for spherical spin glasses in a beautiful series of works by E. Subag. It could be tried here (not easy and not always useful).

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- What about computing quenched complexity?
- One can always try to use the non-rigorous "replicated Kac-Rice" method (Ros-GBA-Biroli-Cammarota 2018).
- But if one wants to be rigorous mathematically, there is only one rather blunt tool: computing higher moments of the number of critical points using an extension of Kac-Rice again. This works only to prove that the quenched complexity is equal to the annealed one! So not when the low temperature phase is FRSB...
- This has been done for spherical spin glasses in a beautiful series of works by E. Subag. It could be tried here (not easy and not always useful).