Very short AFEM history has to include at least

1D results by Babuska et al in the Eighties

Dörfler marking	[Dörfler 1996]
Convergence	[Morin-Nochetto-Siebert 2000]
Optimal rates Poisson problem	[Binev-Dahmen-DeVore 2004]
Optimal rates without coarsening	[Stevenson 2007]
NVB included	[Cascon-Kreuzer-Nochetto-Siebert 2008]
Integral equations and BEMs	[Feischl et al. 2013], [Gantumur 2013]
General boundary conditions	[Aurada et al. 2013]
General 2.-order lin. ellip. PDEs	[Feischl et al. 2014]
AoA	[C-Feischl-Page-Praetorius 2014]
Instance optimality	[Diening-Kreuzer-Stevenson 2015]
AoA separate marking	[C-Rabus 2017]
Inf-Sup stability implies Quasiorthogonality	[Feischl 2022]
Cost Optimality	[Praetorius et al 2021]
Acta Numerica	[Bonito-Canuto-Nochetto-Veeser 2024]

List of Technicalities

Conforming P_1 FEM for Poisson Model Problem (weak form, $H_0^1(\Omega)$, energy scalar product $a(\bullet,\bullet) := \int_\Omega \nabla \bullet \cdot \nabla \bullet \, dx$, energy norm)
Inverse estimates (for polynomials)
Trace inequality (for Sobolev functions)
Discrete trace inequality (for polynomials)
Shape regularity (for triangles, simplices)
Poincaré and Friedrichs inequality (for Sobolev functions)
Equivalence of norms in finite-dimensional vector spaces
Scaling argument (for derivatives of Sobolev functions)
Triangle inequality (in normed linear spaces)
Cauchy inequality (in Hilbert spaces like L^2 or w.r.t. $a(\bullet,\bullet)$)

C, F. Hellwig: Constants in Discrete Poincaré and Friedrichs Inequalities and Discrete Quasi-interpolation, CMAM (arXiv:1709.00577), 2017.

Axioms (A1)–(A4)

Admissible triangulations $\mathbb{T}$ and universal constants $\rho_2 < 1$, $\Lambda_1, \ldots, \Lambda_4$.

(A1)-(A3) hold in 2-level notation for all $\mathcal{T} \in \mathbb{T}$ and $\widehat{\mathcal{T}} \in \mathbb{T}(\mathcal{T})$ with $\eta := \eta_{\mathcal{T}}$, $\widehat{\eta} := \eta_{\widehat{\mathcal{T}}}$, resp. and some $\mathcal{R} \subset \mathcal{T} \wedge |\mathcal{R}| \lesssim |\mathcal{T} \setminus \hat{\mathcal{T}}|$ in (A3)

$$|\widehat{\eta}(\mathcal{T} \cap \hat{\mathcal{T}}) - \eta(\mathcal{T} \cap \hat{\mathcal{T}})| \leq \Lambda_1 \delta(\mathcal{T}, \hat{\mathcal{T}}) \tag{A1}$$

$$\widehat{\eta}(\hat{\mathcal{T}} \setminus \mathcal{T}) \leq \rho_2 \eta(\mathcal{T} \setminus \hat{\mathcal{T}}) + \Lambda_2 \delta(\mathcal{T}, \hat{\mathcal{T}}) \tag{A2}$$

$$\delta(\mathcal{T}, \hat{\mathcal{T}}) \leq \Lambda_3 \eta(\mathcal{R}) \tag{A3}$$

$$\sum_{k=\ell}^{\infty} \delta^2(\mathcal{T}_k, \mathcal{T}_{k+1}) \leq \Lambda_4 \eta_\ell(\mathcal{T}_\ell)^2 \quad \text{for all } \ell \in \mathbb{N}_0 \tag{A4}$$

for the outcome $\mathcal{T}_\ell$ and $\eta_\ell := \eta_{\mathcal{T}_\ell}$ of the adaptive algorithm (AFEM)

Optimality Analysis at a Glance I

- (A12) Estimator reduction $\eta_{\ell+1}^2 \le \rho_\ell \eta_\ell^2 + \Lambda_{12}\delta^2(\mathcal{T}_\ell, \mathcal{T}_{\ell+1})$
- (A12) and (A4) imply convergence from

$$\sum_{k=\ell}^{\infty} \eta_k^2 \lesssim \eta_\ell^2 \quad \text{and then} \quad \sum_{k=0}^{\ell-1} \eta_k^{-1/s} \lesssim \eta_\ell^{-1/s}$$

- (A12) and (A3) imply quasimonotonicity (QM) $\quad \eta(\hat{\mathcal{T}}) \le \Lambda_7 \eta(\mathcal{T})$
- Comparison lemma: $\forall \ell \; \forall 0 < \varrho < 1 \; \exists \hat{\mathcal{T}}_\ell \in \mathbb{T}(\mathcal{T}_\ell) \; \exists \theta_0 < 1$ s.t.

$$\eta(\hat{\mathcal{T}}_\ell) \le \varrho \eta(\mathcal{T}_\ell)$$

$$\varrho \eta_\ell |\mathcal{R}|^s \lesssim \sup_{N \in \mathbb{N}_0} (1+N)^s \min_{\mathcal{T} \in \mathbb{T}(N)} \eta(\mathcal{T}) =: M$$

$$\theta_0 \eta_\ell^2 \le \eta^2(\mathcal{T}_\ell, \mathcal{R}) \text{ for } \mathcal{R} \text{ from (A3) for } \hat{\mathcal{T}}_\ell \in \mathbb{T}(\mathcal{T}_\ell)$$

Optimality Analysis at a Glance II

- $\mathcal{R}$ satisfies bulk criterion if $\theta_A \leq \theta_0$ thus $|\mathcal{M}_\ell^\star| \leq |\mathcal{R}|$ for optimal set $\mathcal{M}_\ell^\star$ of marked cells. AFEM utilizes almost minimal $\mathcal{M}_\ell$, whence

$$|\mathcal{M}_\ell| \lesssim |\mathcal{M}_\ell^\star| \leq |\mathcal{R}|$$

- Set $M := \sup_{N\in\mathbb{N}_0}(1+N)^s \min_{\mathcal{T}\in\mathbb{T}(N)} \eta(\mathcal{T})$ with (writing $\varrho \approx 1$)

$$|\mathcal{R}| \lesssim M^{1/s}\eta_\ell^{-1/s}$$

- Recall closure overhead control and combine with aforementioned estimates for

$$|\mathcal{T}_\ell| - |\mathcal{T}_0| \lesssim \sum_{k=0}^{\ell-1} |\mathcal{M}_k| \lesssim M^{1/s} \sum_{k=0}^{\ell-1} \eta_k^{-1/s} \lesssim M^{1/s}\eta_\ell^{-1/s} \quad \square$$

Comments on AoA

Young persons guide to optimal algorithms and convergence rates

Relations between estimators: Dörfler marking somehow necessary

No local efficiency in AoA lead to optimal convergence rates in terms of estimator

Example adaptive BEM seemingly lack local efficiency!
Reliability in [C-Stephan (1995) Math Comp] with efficiency solely on uniform meshes [C (1996) Math Comp] Despite all this optimal convergence rates in [Vienna UT Team (2013+14) SINUM]

Quasiorthogonality for stable schemes [Feischl (2022) Math Comp]

Cost Optimality [Praetorius et al (2021) Math Comp]

Separate marking for mixed/least squares FEM start at [C-Park (2015) SINUM] and ends collectively in [C-Ma (2021) CAMWA]

Lowest-order adaptive DPG [C-Hellwig (2018) SINUM]