Hybrid high-order methods for convex minimization problems

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Primal problem

- $\Omega \subset \mathbb{R}^n$ bdd. polyhedral L-domain, $2 \le p < \infty$, $1/p + 1/p' = 1$, $f \in L^{p'}(\Omega)$
- $W \in C^1(\mathbb{R}^n)$ is convex and satisfies, for all $a \in \mathbb{R}^n$,

$$c_1|a|^p - c_2 \le W(a) \le c_3|a|^p + c_4$$

$$\text{Minimize } E(v) := \int_\Omega (W(\nabla v) - fv)\,\mathrm{d}x \quad \text{among } v \in V := W_0^{1,p}(\Omega)$$

$$u \in \arg\min E(V) \text{ iff } \int_\Omega \mathrm{D}W(\nabla u) \cdot \nabla v\,\mathrm{d}x = \int_\Omega fv\,\mathrm{d}x$$

Dacorogna: Direct methods in the calculus of variations. Springer (2008)

Dual problem

- $W^*(g) := \sup_{a\in\mathbb{R}^n}(a\cdot g - W(a))$ for $g \in \mathbb{R}^n$
- $a \cdot g \leq W(a) + W^*(g)$ for any $a, g \in \mathbb{R}^n$
- $a \cdot \mathrm{D}W(A) = W(a) + W^*(\mathrm{D}W(a))$ for any $a \in \mathbb{R}^n$
- $\mathcal{Q}(f) := \{\tau \in \Sigma := W^{p'}(\mathrm{div}, \Omega) : \mathrm{div}\,\tau + f = 0 \text{ in } \Omega\}$

$$\text{Maximize } E^*(\tau) := -\int_\Omega W^*(\tau)\,\mathrm{d}x \quad \text{among } \tau \in \mathcal{Q}(f)$$

Duality relations

- Any $v \in V$ and $\tau \in Q(f)$ satisfy

$$\int_\Omega \underbrace{f}_{-\operatorname{div}\tau} v \,\mathrm{d}x = \int_\Omega \underbrace{\tau \cdot \nabla v}_{\leq W(\nabla v) + W^*(\tau)} \mathrm{d}x \leq \int_\Omega (W(\nabla v) + W^*(\tau)) \,\mathrm{d}x$$

$$E^*(\tau) \leq E(v) \quad \forall v \in V, \tau \in Q(f)$$

- Use above argumentation on $u \in \operatorname{arg\,min} E(V)$ and $\sigma := \mathrm{D}W(\nabla u) \in Q(f)$ to obtain

$$u \in \operatorname{arg\,min} E(V) \text{ then } \sigma := \mathrm{D}W(\nabla u) \in \operatorname{arg\,max} E^*(Q(f)) \text{ with } E(u) = E^*(\sigma)$$

Primal-dual gap

- Let $u_C \in V$, $\sigma_h \in Q(f)$ be approximations of u and σ_h

$$E(u_C) - E(u) \leq E(u_C) - E^*(\sigma_h)$$

- Energy error controls other monotonicity errors
- HHO can be used to obtain u_C and σ_h <u>at the same time</u>

Unstabilized HHO method

- $\mathcal{T}$ triangulation into simplices with interior sides $\mathcal{F}(\Omega)$
- $V_h := \mathrm{P}_k(\mathcal{T}) \times \mathrm{P}_k(\mathcal{F}(\Omega))$
- $\Sigma_h := \mathrm{RT}_k^{\mathrm{pw}}(\mathcal{T})$ $(\mathrm{RT}_k(T) := P_k(T)^n + xP_k(T))$

Given $v_h = (v_\mathcal{T}, v_\mathcal{F}) \in V_h$, $\nabla_h v_h \in \Sigma_h$ solve

$$\int_\Omega \nabla_h v_h \cdot \tau_h \,\mathrm{d}x = -\int_\Omega v_\mathcal{T} \operatorname{div}_{\mathrm{pw}} \tau_h \,\mathrm{d}x + \sum_{F \in \mathcal{F}(\Omega)} \int_F \underbrace{v_F}_{=(v_\mathcal{F})|_F} [\tau_h \cdot \nu_F]_F \,\mathrm{d}s \quad \forall \tau_h \in \Sigma_h$$

- $\|\nabla_h \bullet\|_{L^p(\Omega)}$ is a norm in V_h $\to$ no stabilization

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Abbas, Ern, and Pignet: Hybrid high-order methods for finite deformations of hyperelastic materials. Comput. Mech. (2018)

Discrete problem

$$\text{Minimize } E_h(v_h) := \int_\Omega (W(\nabla_h v_h) - f v_\mathcal{T})\,\mathrm{d}x \quad \text{among } v_h = (v_\mathcal{T}, v_\mathcal{F}) \in V_h$$

$$u_h \in \arg\min E_h(V_h) \text{ iff } \int_\Omega \mathrm{D}W(\nabla_h u_h) \cdot \nabla_h v_h\,\mathrm{d}x = \int_\Omega f v_\mathcal{T}\,\mathrm{d}x \ \forall v_h = (v_\mathcal{T}, v_\mathcal{F}) \in V_h$$

$$\sigma_h := \Pi_{\Sigma_h} \mathrm{D}W(\nabla_h u_h) \in Q(f) \text{ if } f \in P_k(\mathcal{T}) \text{ and } u_h \in \arg\min E_h(V_h)$$

Proof of $H(\mathrm{div})$ conformity

Proof.

$$\int_\Omega \underbrace{\mathrm{D}W(\nabla_h u_h)}_{\sigma_h} \cdot \nabla_h v_h \,\mathrm{d}x = \int_\Omega f v_{\mathcal{T}} \,\mathrm{d}x \quad \forall v_h = (v_{\mathcal{T}}, v_{\mathcal{F}}) \in V_h \qquad \text{(dELE)}$$

Fix $F \in \mathcal{F}(\Omega)$, $v_h = (0, v_{\mathcal{F}}) \in V_h$ with $(v_{\mathcal{F}})|_E = 0\ \forall E \in \mathcal{F} \setminus \{F\}$ in (dELE) $\Longrightarrow$

$$0 = \int_\Omega \sigma_h \cdot \nabla_h v_h \,\mathrm{d}x = \cancel{-\int_\Omega v_{\mathcal{T}} \operatorname{div}_{\mathrm{pw}} \sigma_h \,\mathrm{d}x} + \int_F v_F [\sigma_h \cdot \nu_F]_F \,\mathrm{d}s$$

$\Longrightarrow [\sigma_h \cdot \nu_F]_F \perp P_k(F) \Longrightarrow [\sigma_h \cdot \nu_F]_F = 0$ on $F \in \mathcal{F}(\Omega) \Longrightarrow \sigma_h \in \mathrm{RT}_k(\mathcal{T})$

$v_h = (v_{\mathcal{T}}, 0) \in V_h$ in (dELE) $\Longrightarrow \int_\Omega f v_{\mathcal{T}} \,\mathrm{d}x = -\int_\Omega v_{\mathcal{T}} \operatorname{div} \sigma_h \,\mathrm{d}x \Longrightarrow \operatorname{div} \sigma_h + \Pi_{\mathcal{T}}^k f = 0$

Error estimation

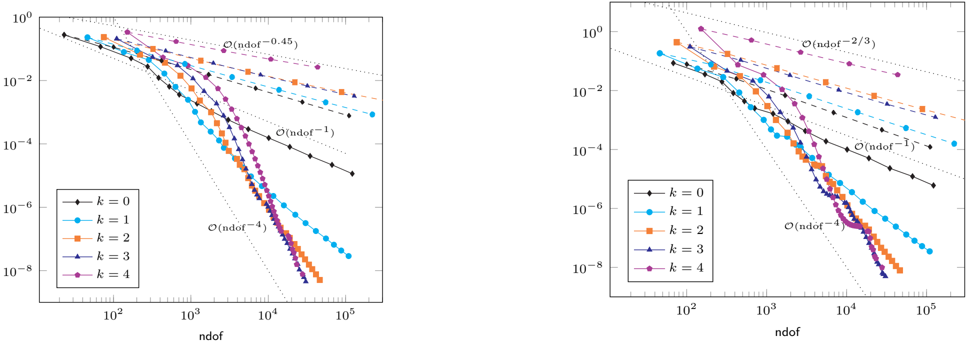
- $\mathcal{R}_h u_h \in P_{k+1}(\mathcal{T})$ solve, for all $p \in P_{k+1}(\mathcal{T})$,
$$\int_\Omega \nabla_{\mathrm{pw}} \mathcal{R}_h u_h \cdot \nabla_{\mathrm{pw}} p \,\mathrm{d}x = -\int_\Omega u_{\mathcal{T}} \Delta_{\mathrm{pw}} p \,\mathrm{d}x + \sum_{F \in \mathcal{F}(\Omega)} \int_F u_F [\nabla_{\mathrm{pw}} p \cdot \nu_F]_F \,\mathrm{d}s,$$
$$\int_T \mathcal{R}_h u_h \,\mathrm{d}x = \int_T u_{\mathcal{T}} \,\mathrm{d}x \;\forall T \in \mathcal{T}$$
- $u_C \in P_{k+1}(\mathcal{T}) \cap V$ is nodal average of $\mathcal{R}_h u_h$

$$E(u_C) - E(u) \leq E(u_C) - E^*(\sigma_h) =: \eta \text{ if } f \in P_k(\mathcal{T})$$

- $0 \leq \eta(T) := \int_T (W(\nabla u_C) - \nabla u_C \cdot \sigma_h + W^*(\sigma_h))$ denotes local refinement indicator

4-Laplace in L-shaped domain [Carstensen Klose 2003]

- $W(a) \coloneqq |a|^4/4$, $u(r,\varphi) = r^{7/8}\sin(7\varphi/8) \in W^{1,4}(\Omega)$
- $f(r,\varphi) = (7/8)^{3/4} r^{-11/8}\sin(7\varphi/8) \in L^{16/11-\varepsilon}(\Omega)$ for any $\varepsilon > 0$



η stress error $\|\sigma - \nabla\Psi(\nabla u_C)\|_{4/3}^2$

Optimal design problem [Kohn Strang 1986, Bartels Carstensen 2008]

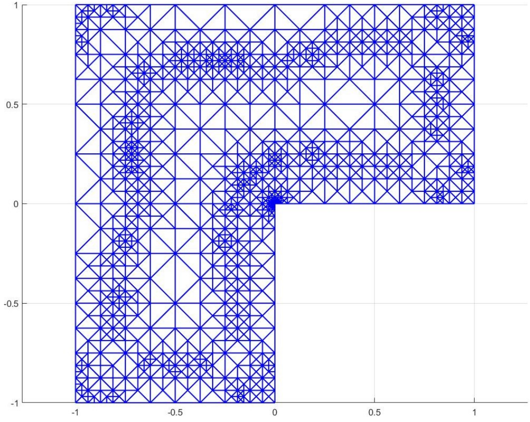
- Given $0 < t_1 < t_2$ and $0 < \mu_1 < \mu_2$ with $t_1\mu_2 = \mu_1 t_2$, define $W(a) := w(|a|)$, $a \in \mathbb{R}^n$, with

$$w(t) := \begin{cases} \mu_2 t^2/2 & \text{if } 0 \le t \le t_1, \\ t_1\mu_2(t - t_1/2) & \text{if } t_1 \le t \le t_2, \\ \mu_1 t^2/2 + t_1\mu_2(t_2/2 - t_1/2) & \text{if } t_2 \le t \end{cases}$$

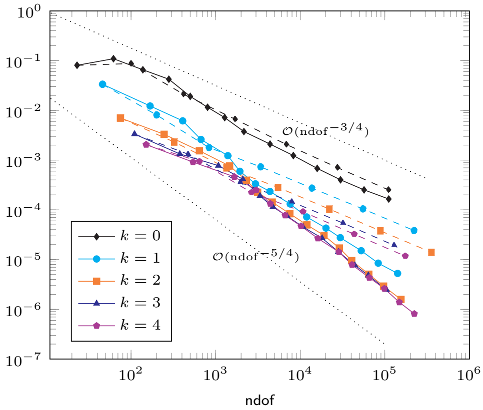
- $\mu_1 = 1$, $\mu_2 = 2$, $t_1 = \sqrt{2\lambda\mu_1/\mu_2}$ for $\lambda = 0.0084$, $t_2 = \mu_2 t_1/\mu_1$ from [Bartels Carstensen 2008]
- $\Omega =$ L-shaped domain, $f \equiv 1$

Material distribution

Optimal design problem



Adaptive triangulation

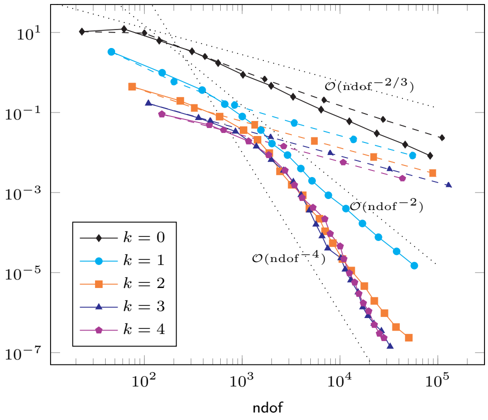


η

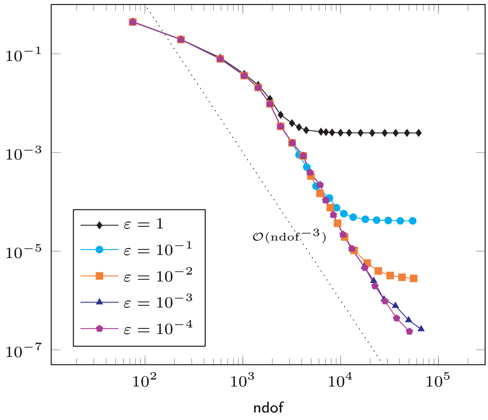
Bingham flow in a pipe

- $W(a) := \mu|a|^2/2 + g|a|$ for any $a \in \mathbb{R}^2$, $g > 0$
- $W^*(\alpha) = \begin{cases} 0 & \text{if } |\alpha| \le g \\ (|\alpha| - g)^2/(2\mu) & \text{if } |\alpha| > g. \end{cases}$
- For the computation of σ_h and u_C, use the regularization
$$W_\varepsilon(a) := \mu|a|^2/2 + g(\sqrt{|a|^2 + \varepsilon^2}) \quad \text{for any } a \in \mathbb{R}^2.$$
- $\Omega =$ L-shaped domain, $f \equiv 10$, $g = 0.2$

Bingham flow in a pipe



η, $\varepsilon = 10^{-4}$



η, $k = 2$